Suggested Solutions for Homework Assignment #1

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. (30 points) Prove that every propositional formula has an equivalent formula in the conjunctive normal form and also an equivalent formula in the disjunctive normal form. (Hint: by induction on the structure of a formula, dealing with both cases simultaneously)

   Solution. To be completed. □

2. (40 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

   (a) $p \vee q \rightarrow r \vdash (p \rightarrow r) \wedge (q \rightarrow r)$
   Solution. To be completed. □

   (b) $\vdash (p \wedge q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$
   Solution.

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3. (30 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

   (a) $\vdash (\neg p \vee q) \rightarrow (p \rightarrow q)$
   Solution. To be completed. □

   (b) $\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$
   Solution.

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