Suggested Solutions for Homework Assignment #4

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. Prove that the following annotated program segments are correct:

   (a) (10 points)

   $\{true\}$
   if $x < y$ **then** $x, y := y, x$ **fi**
   $\{x \geq y\}$

   Solution.

$$\dfrac{\dfrac{\dfrac{\text{pred. calculus + algebra}}{true \wedge x < y \rightarrow y \geq x} \qquad \dfrac{}{\{\ y \geq x\ \}\ x, y := y, x\ \{\ x \geq y\ \}}\text{(Assign)}}{\{\ true \wedge x < y\ \}\ x, y := y, x\ \{\ x \geq y\ \}}\text{(SP)} \qquad \dfrac{\text{pred. calculus + algebra}}{true \wedge \neg(x < y) \rightarrow x \geq y}}{\{\ true\ \}\ \textbf{if}\ x < y\ \textbf{then}\ x, y := y, x\ \textbf{fi}\ \{\ x \geq y\ \}}\text{(If-Then)}$$

□

   (b) (10 points)

   $\{g = 0 \wedge p = n \wedge n \geq 1\}$
   while $p \geq 2$ **do**
   $\quad g, p := g + 1, p - 1$
   od
   $\{g = n - 1\}$

   Solution.

$$\dfrac{\dfrac{\text{pred. calculus + algebra}}{g = 0 \wedge p = n \wedge n = 1 \rightarrow p > 0 \wedge p + g = n} \qquad \alpha \qquad \dfrac{\text{pred. calculus + algebra}}{p > 0 \wedge p + g = n \wedge \neg(p \geq 2) \rightarrow g = n - 1}}{\{\ g = 0 \wedge p = n \wedge n = 1\ \}\ \textbf{while}\ p \geq 2\ \textbf{do}\ g, p := g - 1, p + 1\ \textbf{od}\ \{\ g = n - 1\ \}}\text{(Consequence)}$$

α :

$$\dfrac{\dfrac{\beta \qquad \dfrac{}{\{\ p + 1 > 0 \wedge (p + 1) + (g - 1) = n\ \}\ g, p := g - 1, p + 1\ \{\ p > 0 \wedge p + g = n\ \}}\text{(Assign)}}{\{\ p > 0 \wedge p + g = n \wedge p \geq 2\ \}\ g, p := g - 1, p + 1\ \{\ p > 0 \wedge p + g = n\ \}}\text{(SP)}}{\{\ p > 0 \wedge p + g = n\ \}\ \textbf{while}\ p \geq 2\ \textbf{do}\ g, p := g - 1, p + 1\ \textbf{od}\ \{\ p > 0 \wedge p + g = n \wedge \neg(p \geq 2)\ \}}\text{(while)}$$

β :

$$\dfrac{\text{pred. calculus + algebra}}{p > 0 \wedge p + g = n \wedge p \geq 2 \rightarrow p + 1 > 0 \wedge (p + 1) + (g - 1) = n}$$

□

   (c) (20 points) For this program, prove its total correctness.

   $\{y > 0 \wedge (x \equiv m \pmod{y})\}$
   while $x \geq y$ **do**
   $\quad x := x - y$
   od
   $\{(x \equiv m \pmod{y}) \wedge x < y\}$

Solution.

$$\frac{\alpha \qquad \dfrac{\text{pred. calculus + algebra}}{y > 0 \land (x \equiv m \pmod y) \land \neg(x \ge y) \to (x \equiv m \pmod y) \land x < y}}{\{\ y > 0 \land (x \equiv m \pmod y)\ \}\ \textbf{while}\ x \ge y\ \textbf{do}\ x := x - y\ \textbf{od}\ \{\ (x \equiv m \pmod y) \land x < y\ \}}\ (\text{SP})$$

α :

$$\frac{\beta \qquad \gamma \qquad \dfrac{\text{pred. calculus + algebra}}{y > 0 \land (x \equiv m \pmod y) \land x \ge y \to x \ge 0}}{\begin{array}{c}\{\ y > 0 \land (x \equiv m \pmod y)\ \}\\ \textbf{while}\ x \ge y\ \textbf{do}\ x := x - y\ \textbf{od}\\ \{\ y > 0 \land (x \equiv m \pmod y) \land \neg(x \ge y)\ \}\end{array}}\ (\text{while: simply total})$$

β :

$$\frac{\dfrac{\text{pred. calculus + algebra}}{\begin{array}{c}y > 0 \land (x \equiv m \pmod y) \land x \ge y \to\\ y > 0 \land ((x - y) \equiv m \pmod y)\end{array}} \qquad \dfrac{}{\begin{array}{c}\{\ y > 0 \land ((x - y) \equiv m \pmod y)\ \}\\ x := x - y\\ \{\ y > 0 \land (x \equiv m \pmod y)\ \}\end{array}}\ (\text{Assign})}{\{\ y > 0 \land (x \equiv m \pmod y) \land x \ge y\ \}\ x := x - y\ \{\ y > 0 \land (x \equiv m \pmod y)\ \}}\ (\text{SP})$$

γ :

$$\frac{\dfrac{\text{pred. calculus + algebra}}{y > 0 \land (x \equiv m \pmod y) \land x \ge y \land x = Z \to x - y < Z} \qquad \dfrac{}{\{\ x - y < Z\ \}\ x := x - y\ \{\ x < Z\ \}}\ (\text{Assign})}{\{\ y > 0 \land (x \equiv m \pmod y) \land x \ge y \land x = Z\ \}\ x := x - y\ \{\ x < Z\ \}}\ (\text{SP})$$

□