Suggested Solutions for Homework Assignment #2

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. (20 points) Please provide a precise description, using formulae in first-order logic, for each of the following requirements. The functions/constants and predicates you may use are: $+$, $\times$, 0, 1, 2, $<$, $=$, $\leq$, plus those introduced in the requirement statements. Make assumptions where you see necessary.

   (a) The array $A[0..N-1]$ (of integers) represents a max heap with $A[0]$ as the root.
   Solution. $\forall i(0 \leq i \leq N-1 \rightarrow ((2\times i+1 \leq N-1 \rightarrow A[i] \geq A[2\times i+1]) \wedge (2\times i+2 \leq N-1 \rightarrow A[i] \geq A[2\times i+2]))$ □

   (b) The array $A[0..N-1]$ (of integers) is cyclically sorted in an increasing order. (Note: $3, 4, 0, 1, 2$, for example, is a cyclically sorted list of integers.)
   Solution.

$$\begin{array}{l} \forall i(0 \leq i < N-1 \rightarrow A[i] \leq A[i+1]) \\ \vee \\ ((A[N-1] \leq A[0])\wedge \\ \ \exists j((0 < j \leq N-1)\wedge \\ \quad\quad \forall i(0 \leq i < j-1 \rightarrow A[i] \leq A[i+1])\wedge \\ \quad\quad \forall i(j \leq i < N-1 \rightarrow A[i] \leq A[i+1]))) \end{array}$$

□

2. (20 points) Prove, using *Natural Deduction*, the validity of the following sequents:

   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall xP(x) \rightarrow \forall xQ(x)$
   Solution. Assume w does not occur free either in $P(x)$ or in $Q(x)$.

$$\dfrac{\alpha \qquad \dfrac{\dfrac{}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xP(x)}(Hyp)}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash P(w)}(\forall E)}{\dfrac{\dfrac{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash Q(w)}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x)}(\forall I)}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall xP(x) \rightarrow \forall xQ(x)}(\rightarrow I)}(\rightarrow E)$$

α :

$$\dfrac{\dfrac{}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall x(P(x) \rightarrow Q(x))}(Hyp)}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash P(w) \rightarrow Q(w)}(\forall E)$$

□

   (b) $\vdash \exists x\forall yP(x, y) \rightarrow \forall y\exists xP(x, y)$
   Solution. Assume both w and z do not occur free in $P(x, y)$.

$$
\dfrac{
\dfrac{}{\exists x \forall y P(x,y) \vdash \exists x \forall y P(x,y)}\,(Hyp)
\qquad
\dfrac{\dfrac{\dfrac{}{\exists x \forall y P(x,y), \forall y P(z,y) \vdash \forall y P(z,y)}\,(Hyp)}{\exists x \forall y P(x,y), \forall y P(z,y) \vdash P(z,w)}\,(\forall E)}{\exists x \forall y P(x,y), \forall y P(z,y) \vdash \exists x P(x,w)}\,(\exists I)
}{
\dfrac{\dfrac{\exists x \forall y P(x,y) \vdash \exists x P(x,w)}{\exists x \forall y P(x,y) \vdash \forall y \exists x P(x,y)}\,(\forall I)}{\vdash \exists x \forall y P(x,y) \to \forall y \exists x P(x,y)}\,(\to I)
}\,(\exists E)
$$

□

3. (20 points) Prove, using *Natural Deduction* for the first-order logic with equality (=), that = is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity), which follows from the =-Introduction rule.

   (a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry)
   Solution.

$$
\dfrac{\dfrac{}{t_2 = t_1 \vdash t_2 = t_1}\,(Hyp) \qquad \dfrac{}{t_2 = t_1 \vdash t_2 = t_2}\,(= I)}{t_2 = t_1 \vdash t_1 = t_2}\,(= E)
$$

□

   (b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity)
   Solution.

$$
\dfrac{\dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_2 = t_3}\,(Hyp) \qquad \dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_2}\,(Hyp)}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,(= E)
$$

□

4. (20 points) Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality.

   (a) $\dfrac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,(= \mathit{Symmetry})$
   Solution.

$$
\dfrac{\dfrac{\dfrac{}{\Gamma, t_2 = t_1 \vdash t_1 = t_2}\,(Axiom(Symmetry))}{\Gamma \vdash t_2 = t_1 \to t_1 = t_2}\,(\to I) \qquad \Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,(\to E)
$$

□

   (b) $\dfrac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,(= \mathit{Transitivity})$
   Solution.

$$
\dfrac{\dfrac{\alpha \qquad \Gamma \vdash t_1 = t_2}{\Gamma \vdash t_2 = t_3 \to t_1 = t_3}\,(\to E) \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,(\to E)
$$

   α :

$$
\dfrac{\dfrac{\dfrac{}{\Gamma, t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,(Axiom(Transitivity))}{\Gamma, t_1 = t_2 \vdash t_2 = t_3 \to t_1 = t_3}\,(\to I)}{\Gamma \vdash t_1 = t_2 \to (t_2 = t_3 \to t_1 = t_3)}\,(\to I)
$$

$\square$

5. (20 points) A first-order theory for *groups* contains the following three axioms:

   - $\forall a \forall b \forall c(a \cdot (b \cdot c) = (a \cdot b) \cdot c)$. (Associativity)
   - $\forall a((a \cdot e = a) \wedge (e \cdot a = a))$. (Identity)
   - $\forall a((a \cdot a^{-1} = e) \wedge (a^{-1} \cdot a = e))$. (Inverse)

   Here $\cdot$ is the binary operation, e is a constant, called the identity, and $(\cdot)^{-1}$ is the inverse function which gives the inverse of an element. Let M denote the set of the three axioms. Prove, using *Natural Deduction* plus the derived rules in the preceding problem, the validity of the following sequent:

   $M \vdash \forall a \forall b \forall c((a \cdot b = a \cdot c) \to b = c)$.

   (Hint: a typical proof in algebra books is the following: $b = e \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c) = (a^{-1} \cdot a) \cdot c = e \cdot c = c$.)

   Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \qquad \delta}{M, x \cdot y = x \cdot z \vdash y = z}\,(=E)}{M \vdash (x \cdot y = x \cdot z) \to y = z}\,(\to I)}{M \vdash \forall c((x \cdot y = x \cdot c) \to y = c)}\,(\forall I)}{M \vdash \forall b \forall c((x \cdot b = x \cdot c) \to b = c)}\,(\forall I)}{M \vdash \forall a \forall b \forall c((a \cdot b = a \cdot c) \to b = c)}\,(\forall I)$$

α :

$$\dfrac{\dfrac{\beta \qquad \gamma}{M, x \cdot y = x \cdot z \vdash (x^{-1} \cdot x) \cdot y = y}\,(=E) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash \forall a \forall b \forall c(a \cdot (b \cdot c) = (a \cdot b) \cdot c)}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash \forall b \forall c(x^{-1} \cdot (b \cdot c) = (x^{-1} \cdot b) \cdot c)}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash \forall c(x^{-1} \cdot (x \cdot c) = (x^{-1} \cdot x) \cdot c)}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) \cdot y}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot y) = y}\,(=E)$$

β :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash \forall a(a \cdot a^{-1} = e \wedge a^{-1} \cdot a = e)}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot x = e}\,(\wedge E_2)}{M, x \cdot y = x \cdot z \vdash e = x^{-1} \cdot x}\,(= Symmetry)$$

γ :

$$\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash \forall a(a \cdot e = a \wedge e \cdot a = a)}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash y \cdot e = y \wedge e \cdot y = y}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash e \cdot y = y}\,(\wedge E_2)$$

δ :

$$\dfrac{\dfrac{\overline{M, x \cdot y = x \cdot z \vdash x \cdot y = x \cdot z}\ (Hyp)}{M, x \cdot y = x \cdot z \vdash x \cdot z = x \cdot y}\ (= Symmetry) \qquad \dfrac{\text{the proof tree is similar to } \alpha}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot z) = z}}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot y) = z}\ (= E)$$

□