# Natural Deduction in the Sequent Form (for Intuitionistic Logic)

## 1 Deduction Rules

$$\frac{}{A_1, \cdots, A_i, \cdots, A_n \vdash A_i}\ (Hyp^i)$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge I) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge E_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge E_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee I_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee I_2) \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee E)$$

$$\frac{}{\Gamma \vdash \top}\ (\top I)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to I) \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ (\to E)$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot E)$$

The above rule says that, if we can deduce $\bot$ (representing a contradiction), then we can deduce anything. $\neg A$ is then represented (i.e., defined) as $A \to \bot$. With the $\to$-elimination rule, one can deduce a contradiction (and hence anything) from $\neg A$ and $A$.

$$\frac{\Gamma \vdash B[y/x]}{\Gamma \vdash \forall x B}\ (\forall I) \qquad \frac{\Gamma \vdash \forall x B}{\Gamma \vdash B[t/x]}\ (\forall E)$$

$$\frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x B}\ (\exists I) \qquad \frac{\Gamma \vdash \exists x B \qquad \Gamma, B[y/x] \vdash C}{\Gamma \vdash C}\ (\exists E)$$

In the rules above, we assume that all substitutions are admissible. In Rule $\forall I$, $y$ does not occur free in $\Gamma$ or $B$, and in Rule $\exists E$, $y$ does not occur free in $\Gamma$, $B$, or $C$.

## 2 Deduction Rules with Proof Terms

$$\frac{}{x_1 : A_1, \cdots, x_i : A_i, \cdots, x_n : A_n \vdash x_i : A_i}\ (Hyp^i)$$

$$\frac{\Gamma \vdash t_1 : A \qquad \Gamma \vdash t_2 : B}{\Gamma \vdash (t_1, t_2) : A \wedge B}\ (\wedge I) \qquad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \mathbf{fst}(t) : A}\ (\wedge E_1) \qquad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \mathbf{snd}(t) : B}\ (\wedge E_2)$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathbf{inl}(B, t) : A \vee B}\ (\vee I_1) \qquad \frac{\Gamma \vdash t : B}{\Gamma \vdash \mathbf{inr}(A, t) : A \vee B}\ (\vee I_2)$$

$$\frac{\Gamma \vdash t : A \vee B \qquad \Gamma, x : A \vdash t_1 : C \qquad \Gamma, y : B \vdash t_2 : C}{\Gamma \vdash \mathbf{case}(t, x.t_1, y.t_2) : C}\ (\vee E)$$

$$\frac{}{\Gamma \vdash \mathbf{unit} : \top}\ (\top I)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash (\lambda x : A.t) : A \to B}\ (\to I) \qquad \frac{\Gamma \vdash t_1 : A \to B \qquad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1\ t_2 : B}\ (\to E)$$

In the $(\to I)$ rule above, it is assumed that $B$ does not contain $x$ (so $B$ is independent of $A$).

$$\frac{\Gamma \vdash t : \bot}{\Gamma \vdash \mathbf{abort}(A, t) : A}\ (\bot E)$$

$$\frac{\Gamma, y : A \vdash t : B[y/x]}{\Gamma \vdash (\lambda x : A.t) : \forall x : A, B}\ (\forall I) \qquad \frac{\Gamma \vdash t_1 : \forall x : A, B \qquad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1\ t_2 : B[t_2/x]}\ (\forall E)$$

$$\frac{\Gamma \vdash t_1 : A \qquad \Gamma \vdash t_2 : B[t_1/x]}{\Gamma \vdash (t_1, t_2) : \exists x : A, B}\ (\exists I) \qquad \frac{\Gamma \vdash t_1 : \exists x : A, B \qquad \Gamma, y : A, z : B[y/x] \vdash t_2 : C}{\Gamma \vdash \mathbf{open}(t_1, y.z.t_2) : C}\ (\exists E)$$

In Rule $\forall I$, $y$ does not occur free in $\Gamma$ or $B$, and in Rule $\exists E$, $y$ does not occur free in $\Gamma$, $B$, or $C$; $\mathbf{open}((t_1, t_2), y.z.t) \triangleq t[t_2/z][t_1/y]$.