

Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Let $A = \{a, b, c, d, e, f\}$ and $R = \{(a, c), (b, c), (d, e)\}$ (which is a binary relation on A).
 - (a) Give a symmetric and transitive but not reflexive binary relation on A that includes R . Please present the relation using a directed graph.
 - (b) Find the smallest equivalence relation on A that includes R . Please present the relation using a directed graph.
2. Let $L = \{w \in \{0, 1\}^* \mid w \text{ contains } 011 \text{ as a substring or ends with a } 0\}$.
 - (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes L . The fewer states your NFA has, the more points you will be credited for this problem.
 - (b) Convert the preceding NFA systematically into an equivalent DFA (using the procedure discussed in class). Do not attempt to optimize the number of states, though you may omit the unreachable states.
3. Let $L = \{w \in \{0, 1\}^* \mid w \text{ does not contain } 011 \text{ or } 100 \text{ as a substring}\}$.
 - (a) Draw the state diagram of a DFA, with as few states as possible, that recognizes L . The fewer states your DFA has, the more points you will be credited for this problem.
 - (b) Translate the DFA in (a) systematically to an equivalent context-free grammar (using the procedure discussed in class).
4. Let $L = \{1^p \mid p \text{ is a prime number less than } 2^{2^{10}}\}$. Is L a regular language? Why or why not?
5. Let $C_n = \{x \mid x \text{ is a binary number that is a multiple of } n\}$. Show that for each $n \geq 1$, the language C_n is regular.
6. A *synchronizing sequence* for a DFA $M = (Q, \Sigma, \delta, q_0, F)$ and some “home” state $h \in Q$ is a string $s \in \Sigma^*$ such that, for every $q \in Q$, $\delta(q, s) = h$. A DFA is said to be *synchronizable* if it has a synchronizing sequence for some state. Try to find a 5-state synchronizable DFA with a synchronizing sequence as long as possible. What is the longest synchronizing sequence for the DFA and which state is the

corresponding home state? The longer the synchronizing sequence is, the more points you will be credited for this problem.

7. Draw the state diagram of a pushdown automaton (PDA) that recognizes the following language: $\{w \in \{a, b, c\}^* \mid \text{the number of } a\text{'s in } w \text{ equals that of } b\text{'s or } c\text{'s}\}$. Please make the PDA as simple as possible and explain the intuition behind the PDA.
8. Let $A/B = \{w \mid wx \in A \text{ for some } x \in B\}$. Show that, if A is context free and B is regular, then A/B is context free.
9. Show that, if G is a CFG in Chomsky normal form, then any string $w \in L(G)$ of length $n \geq 1$, exactly $2n - 1$ steps are required for any derivation of w .
10. Prove that the language over $\{a, b, c\}$ with equal numbers of a 's, b 's, and c 's is not context free.

Appendix

- Properties of a binary relation R on A :
 - R is *reflexive* if for every $x \in A$, xRx .
 - R is *symmetric* if for every $x, y \in A$, xRy if and only if yRx .
 - R is *transitive* if for every $x, y, z \in A$, xRy and yRz implies xRz .
- A context-free grammar is in **Chomsky normal form** if every rule is of the form

$$\begin{aligned} A &\rightarrow BC \text{ or} \\ A &\rightarrow a \end{aligned}$$

where a is any terminal and A , B , and C are any variables—except that B and C may not be the start variable. In addition,

$$S \rightarrow \varepsilon$$

is permitted if S is the start variable.

- (Pumping Lemma for Context-Free Languages) If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions: (1) for each $i \geq 0$, $uv^i xy^i z \in A$, (2) $|vy| > 0$, and (3) $|vxy| \leq p$.
- The language $\{a^n b^n c^n \mid n \geq 0\}$ is not context free.