Theory of Computing

Introduction and Preliminaries
(Based on [Sipser 2006, 2013])

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What It Is

- The central question:

  What are the fundamental capabilities and limitations of computers?

- Three main areas:
  - *Automata Theory*
  - *Computability Theory*
  - *Complexity Theory*

Complexity Theory

- Some problems are easy and some hard.
  For example, sorting is easy and scheduling is hard.
- The central question of complexity theory:
  What makes some problems computationally hard and others easy?
- We don't have the answer to it.
- However, researchers have found a scheme for classifying problems according to their computational difficulty.
- One practical application: cryptography/security.

Dealing with Hard Problems

Options for dealing with a hard problem:

- Try to simplify it (the hard part of the problem might be unnecessary).
- Settle for an approximate solution.
- Find a solution that usually runs fast.
- Consider alternative types of computation.

Computability Theory

- Alan Turing, among other mathematicians, discovered in the 1930s that certain basic problems cannot be solved by computers.
- One example is the problem of determining whether a mathematical statement is true or false.
- Theoretical models of computers developed at that time eventually lead to the construction of actual computers.
- The theories of computability and complexity are closely related.
- *Complexity theory* seeks to classify problems as easy ones and hard ones, while in *computability theory* the classification is by whether the problem is solvable or not.

Automata Theory

- The theories of computability and complexity require a precise, formal definition of a *computer*.
- *Automata theory* deals with the definitions and properties of mathematical models of computation.
- Two basic and practically useful models:
  - *Finite-state*, or simply *finite*, *automaton*
  - *Context-free grammar* (pushdown automaton)

Sets

- Set, element (member), subset, proper subset
- Multiset
- Description of a set
- The empty set ($\emptyset$)
- Finite set, infinite set
- Union, intersection, complement
- Power set
- Venn diagram

Sets (cont.)

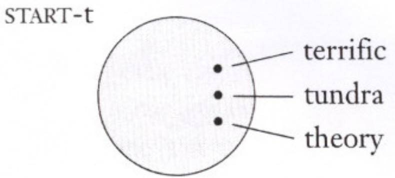


FIGURE 0.1
Venn diagram for the set of English words starting with "t"

Source: [Sipser 2006]

Sets (cont.)

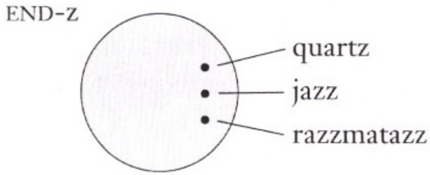


FIGURE 0.2
Venn diagram for the set of English words ending with "z"

Source: [Sipser 2006]

Sets (cont.)

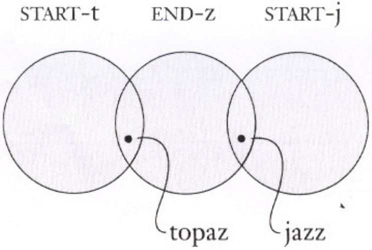


FIGURE 0.3
Overlapping circles indicate common elements

Source: [Sipser 2006]

Sets (cont.)

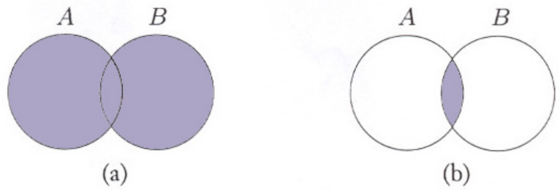


FIGURE 0.4
Diagrams for (a) $A \cup B$ and (b) $A \cap B$

Source: [Sipser 2006]

Sequences and Tuples

- A *sequence* of objects is a list of these objects in some order. Order is essential and repetition is also allowed.
- Finite sequences are often called *tuples*. A sequence with k elements is a k-*tuple*; a 2-tuple is also called a *pair*.
- The *Cartesian product*, or cross product, of A and B, written as $A \times B$, is the set of all pairs (x, y) such that $x \in A$ and $y \in B$.
- Cartesian products generalize to k sets, A_1, A_2, ..., A_k, written as $A_1 \times A_2 \times \ldots \times A_k$. A^k is a shorthand for $A \times A \times \ldots \times A$ (k times).

Functions

- A *function* sets up an *input-output* relationship, where the same input always produces the same output.
- If f is a function whose output is b when the input is a, we write $f(a) = b$.
- A function is also called a *mapping*; if $f(a) = b$, we say that f maps a to b.

Functions (cont.)

- The set of possible inputs to a function is called its *domain*; the outputs come from a set called its *range*.
- A function is *onto* if it uses all the elements of the range (it is *one-to-one* if ...).
- The notation $f : D \longrightarrow R$ says that f is a function with domain D and range R.
- More notions and terms: *k-ary function*, *unary function*, *binary function*, *infix notation*, *prefix notation*

Relations

- A *predicate*, or property, is a function whose range is $\{\text{TRUE,FALSE}\}$.
- A predicate whose domain is a set of k-tuples $A \times \ldots \times A$ is called a (k-ary) *relation* on A.
- A 2-ary relation is also called a *binary relation*.

Equivalence Relations

- An *equivalence relation* is a special type of binary relation that captures the notion of two objects being *equal* in some sense.
- A binary relation R on A is an equivalence relation if
  1. R is *reflexive* (for every x in A, xRx),
  2. R is *symmetric* (for every x and y in A, xRy if and only if yRx), and
  3. R is *transitive* (for every x, y, and z in A, xRy and yRz implies xRz).

Graphs

- Undirected graph, node (vertex), edge (link), degree
- Description of a graph: $G = (V, E)$
- Labeled graph
- Subgraph, induced subgraph
- Path, simple path, cycle, simple cycle
- Connected graph
- Tree, root, leaf
- Directed graph, outdegree, indegree
- Strongly connected graph

Graphs (cont.)

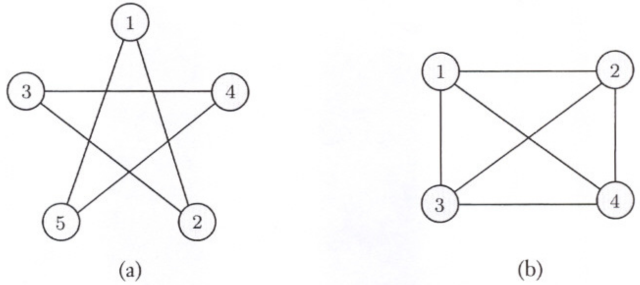


FIGURE **0.12**
Examples of graphs

Source: [Sipser 2006]

Graphs (cont.)

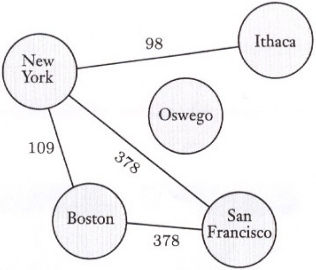


FIGURE 0.13
Cheapest nonstop air fares between various cities

Source: [Sipser 2006]

Graphs (cont.)

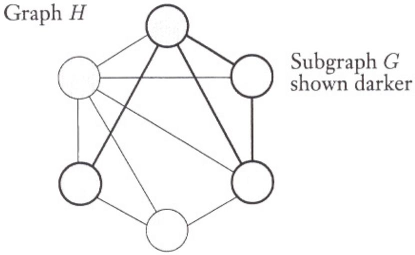


FIGURE 0.14
Graph G (shown darker) is a subgraph of H

Source: [Sipser 2006]

Graphs (cont.)

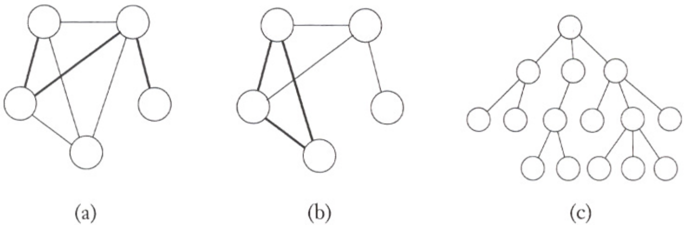


FIGURE 0.15
(a) A path in a graph, (b) a cycle in a graph, and (c) a tree

Source: [Sipser 2006]

Graphs (cont.)

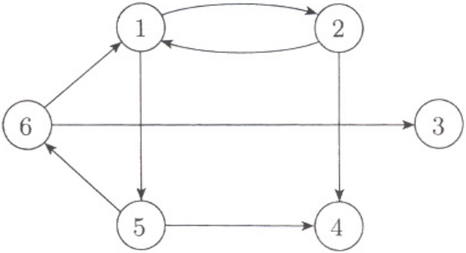


FIGURE 0.16
A directed graph

Source: [Sipser 2006]

Graphs (cont.)

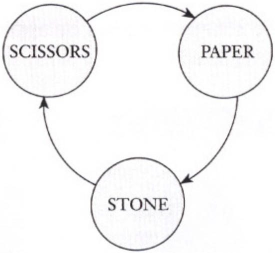


FIGURE 0.18
The graph of the relation *beats*

Source: [Sipser 2006]

Strings and Languages

- An *alphabet* is any finite set of *symbols*.
- A *string* over an alphabet is a finite sequence of symbols from that alphabet.
- The *length* of a string w, written as $|w|$, is the number of symbols that w contains.
- The string of length 0 is called the *empty string*, written as ε.
- The *concatenation* of x and y, written as xy, is the string obtained from appending y to the end of x.
- A *language* is a set of strings.
- More notions and terms: *reverse*, *substring*, *lexicographic ordering*.

Boolean Logic

- *Boolean logic* is a mathematical system built around the two *Boolean values* TRUE (1) and FALSE (0).
- Boolean values can be manipulated with *Boolean operations*: *negation* or NOT ($\neg$), *conjunction* or AND ($\wedge$), *disjunction* or OR ($\vee$).

$$
\begin{array}{lll}
0 \wedge 0 \triangleq 0 & 0 \vee 0 \triangleq 0 & \neg 0 \triangleq 1 \\
0 \wedge 1 \triangleq 0 & 0 \vee 1 \triangleq 1 & \neg 1 \triangleq 0 \\
1 \wedge 0 \triangleq 0 & 1 \vee 0 \triangleq 1 & \\
1 \wedge 1 \triangleq 1 & 1 \vee 1 \triangleq 1 &
\end{array}
$$

- Unknown Boolean values are represented symbolically by *Boolean variables* or *propositions*, e.g., P, Q, etc.

Boolean Logic (cont.)

- Additional Boolean operations: *exclusive or* or XOR ($\oplus$), *equality/equivalence* ($\leftrightarrow$ or $\equiv$), *implication* ($\rightarrow$).

$$
\begin{array}{ccc}
0 \oplus 0 \triangleq 0 & 0 \leftrightarrow 0 \triangleq 1 & 0 \rightarrow 0 \triangleq 1 \\
0 \oplus 1 \triangleq 1 & 0 \leftrightarrow 1 \triangleq 0 & 0 \rightarrow 1 \triangleq 1 \\
1 \oplus 0 \triangleq 1 & 1 \leftrightarrow 0 \triangleq 0 & 1 \rightarrow 0 \triangleq 0 \\
1 \oplus 1 \triangleq 0 & 1 \leftrightarrow 1 \triangleq 1 & 1 \rightarrow 1 \triangleq 1
\end{array}
$$

- All in terms of conjunction and negation:

$$
\begin{array}{rcl}
P \vee Q & \equiv & \neg(\neg P \wedge \neg Q) \\
P \rightarrow Q & \equiv & \neg P \vee Q \\
P \leftrightarrow Q & \equiv & (P \rightarrow Q) \wedge (Q \rightarrow P) \\
P \oplus Q & \equiv & \neg(P \leftrightarrow Q)
\end{array}
$$

Logical Equivalences and Laws

- Two logical expressions/formulae are *equivalent* if each of them implies the other, i.e., they have the same truth value.
- Equivalence plays a role analogous to equality in algebra.
- Some laws of Boolean logic:
  - (Distributive) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
  - (Distributive) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  - (De Morgan's) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
  - (De Morgan's) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Definitions, Theorems, and Proofs

- *Definitions* describe the objects and notions that we use. Precision is essential to any definition.
- After we have defined various objects and notions, we usually make *mathematical statements* about them. Again, the statements must be precise.
- A *proof* is a convincing logical argument that a statement is true. The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof.
- A *theorem* is a mathematical statement proven true. *Lemmas* are proven statements for assisting the proof of another more significant statement.
- *Corollaries* are statements seen to follow easily from other proven ones.

Finding Proofs

- Find proofs isn't always easy; no one has a recipe for it.
- Below are some helpful general strategies:
  1. Carefully read the statement you want to prove.
  2. Rewrite the statement in your own words.
  3. Break it down and consider each part separately.
     For example, $P \iff Q$ consists of two parts: $P \rightarrow Q$ (the forward direction) and $Q \rightarrow P$ (the reverse direction).
  4. Try to get an intuitive feeling of why it should be true.

Tips for Producing a Proof

- A well-written proof is a sequence of statements, wherein each one follows by simple reasoning from previous statements in the sequence.
- Tips for producing a proof:
  - *Be patient*. Finding proofs takes time.
  - *Come back to it*. Look over the statement, think about it, leave it, and then return some time later.
  - *Be neat*. Use simple, clear text and/or pictures; make it easy for others to understand.
  - *Be concise*. Emphasize high-level ideas, but be sure to include enough details of reasoning.

An Example Proof

Theorem

For any two sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. We show that every element of $\overline{A \cup B}$ is also an element of $\overline{A} \cap \overline{B}$ and vice versa.

Forward ($x \in \overline{A \cup B} \to x \in \overline{A} \cap \overline{B}$):

$$
\begin{array}{lll}
 & x \in \overline{A \cup B} & \\
\to & x \notin A \cup B & \text{, def. of complement} \\
\to & x \notin A \text{ and } x \notin B & \text{, def. of union} \\
\to & x \in \overline{A} \text{ and } x \in \overline{B} & \text{, def. of complement} \\
\to & x \in \overline{A} \cap \overline{B} & \text{, def. of intersection}
\end{array}
$$

Reverse ($x \in \overline{A} \cap \overline{B} \to x \in \overline{A \cup B}$): ...

Another Example Proof

IM NTU

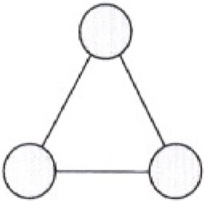
Theorem

In any graph G, the sum of the degrees of the nodes of G is an even number.

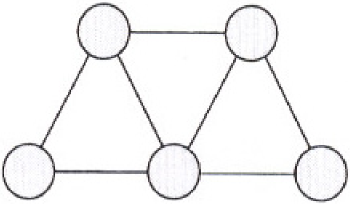
Proof.

- Every edge in G connects two nodes, contributing 1 to the degree of each.
- Therefore, each edge contributes 2 to the sum of the degrees of all the nodes.
- If G has e edges, then the sum of the degrees of the nodes of G is $2e$, which is even.

Another Example Proof (cont.)



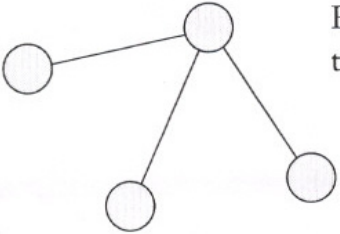
$$\text{sum} = 2+2+2 = 6$$



$$\text{sum} = 2+3+4+3+2 = 14$$

Source: [Sipser 2006]

Another Example Proof (cont.)



Every time an edge is added, the sum increases by 2.

Source: [Sipser 2006]

Types of Proof

- *Proof by construction*:
  prove that a particular type of object exists, by showing how to construct the object.
- *Proof by contradiction*:
  prove a statement by first assuming that the statement is false and then showing that the assumption leads to an obviously false consequence, called a contradiction.
- *Proof by induction*:
  prove that all elements of an infinite set have a specified property, by exploiting the inductive structure of the set.

Proof by Construction

IM NTU

Theorem

For each even number n greater than 2, there exists a 3-regular graph with n nodes.

Proof. Construct a graph $G = (V, E)$ with $n \ (= 2k \geq 2)$ nodes as follows.

Let V be $\{0, 1, \ldots, n-1\}$ and E be defined as

$$
\begin{aligned}
E &= \{\{i, i+1\} \mid \text{for } 0 \leq i \leq n-2\} \cup \\
&\quad \{\{n-1, 0\}\} \cup \\
&\quad \{\{i, i+n/2\} \mid \text{for } 0 \leq i \leq n/2 - 1\}.
\end{aligned}
$$

Proof by Contradiction

IM NTU

Theorem

$\sqrt{2}$ is irrational.

Proof. Assume toward a contradiction that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = \frac{m}{n}$ for some integers m and n, which *cannot both be even*.

$$
\begin{array}{ll}
\sqrt{2} = \frac{m}{n} & \text{, from the assumption} \\
n\sqrt{2} = m & \text{, multipl. both sides by } n \\
2n^2 = m^2 & \text{, square both sides} \\
m \text{ is even} & \text{, } m^2 \text{ is even} \\
2n^2 = (2k)^2 = 4k^2 & \text{, from the above two} \\
n^2 = 2k^2 & \text{, divide both sides by 2} \\
n \text{ is even} & \text{, } n^2 \text{ is even}
\end{array}
$$

Now both m and n are even, a contradiction.

Example: Home Mortgages

P: the *principle* (amount of the original loan).
I: the yearly *interest rate*.
Y: the monthly payment.
M: the *monthly* multiplier $= 1 + I/12$.
P_t: the amount of loan outstanding after the t-th month; $P_0 = P$ and $P_{k+1} = P_k M - Y$.

Theorem

For each $t \geq 0$,

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right).$$

Proof by Induction

Theorem

For each $t \geq 0$*,*

$$P_t = PM^t - Y(\frac{M^t - 1}{M - 1}).$$

Proof. The proof is by induction on t.

- *Basis*: When $t = 0$, $PM^0 - Y(\frac{M^0 - 1}{M - 1}) = P = P_0$.

Proof by Induction (cont.)

- *Induction step*: When $t = k + 1$ ($k \geq 0$),

$$
\begin{array}{rl}
 & P_{k+1} \\
= & \quad \{\text{definition of } P_t\} \\
 & P_k M - Y \\
= & \quad \{\text{the induction hypothesis}\} \\
 & (PM^k - Y(\frac{M^k - 1}{M-1}))M - Y \\
= & \quad \{\text{distribute } M \text{ and rewrite } Y\} \\
 & PM^{k+1} - Y(\frac{M^{k+1} - M}{M-1}) - Y(\frac{M-1}{M-1}) \\
= & \quad \{\text{combine the last two terms}\} \\
 & PM^{k+1} - Y(\frac{M^{k+1} - 1}{M-1})
\end{array}
$$