Turing Machines
(Based on [Sipser 2006, 2013])

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Turing Machines

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- Finite and pushdown automata are too restricted to serve as models of general-purpose computers.
- A *Turing machine* is similar to a finite automaton but with an unlimited and unrestricted memory—an infinite tape. It has a tape head that can read and write symbols and move around on the tape.
- A Turing machine can do everything that a real computer (as we know it) can do.
- Nonetheless, there are problems that no Turing machines, and hence no real computers, can solve.

Turing Machines (cont.)

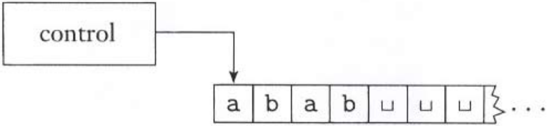

FIGURE **3.1**
Schematic of a Turing machine

Source: [Sipser 2006]

An Example Turing Machine

Let $B = \{w\#w \mid w \in \{0,1\}^*\}$. A Turing machine M_1 for B may work as follows:

1. Scan the input to be sure that it contains a single $\#$ symbol. If not, *reject*.
2. Zig-zag across the tape to corresponding positions on either side of the $\#$ symbol to check whether these positions contain the same symbol. If they do not, *reject*.
   Cross off symbols as they are checked.
3. When all symbols to the left of $\#$ have been crossed off, check for any remaining symbols to the right of the $\#$. If any symbols remain, *reject*; otherwise, *accept*.

An Example Turing Machine (cont.)

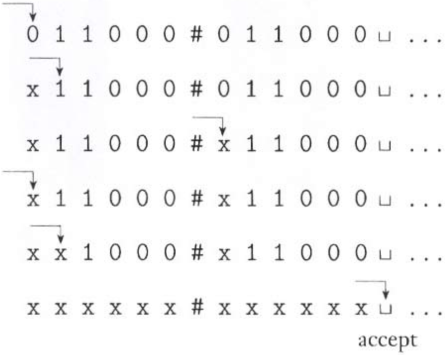

FIGURE 3.2
Snapshots of Turing machine M_1 computing on input 011000#011000

Source: [Sipser 2006]

Formal Definition of a TM

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Definition (3.3)

A **Turing machine** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{accept}}, q_{\mathrm{reject}})$, where Q, Σ, and Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet, where the *blank* symbol $\sqcup \notin \Sigma$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\mathrm{accept}} \in Q$ is the accept state, and
7. $q_{\mathrm{reject}} \in Q$ is the reject state.

Configurations of a TM

- As a TM computes, changes occur in
  1. the current state,
  2. the current tape contents, and
  3. the current head location.
- A setting of these three items is called a **configuration** of the TM.
- We write uqv to denote the configuration where
  1. the current state is q,
  2. the current tape contents is uv, and
  3. the current head location is the first symbol of v.

  (The tape contains only blanks following the last symbol of v.)

Configurations of a TM (cont.)

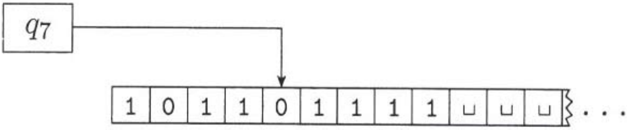

FIGURE 3.4
A Turing machine with configuration $1011q_701111$

Source: [Sipser 2006]

Configurations of a TM (cont.)

- $q_0 w$ is the *start configuration* on input w.
- $u q_{\mathrm{accept}} v$ is an *accepting configuration*.
- $u q_{\mathrm{reject}} v$ is a *rejecting configuration*.
- Accepting and rejecting configurations are *halting configurations*.

Computation of a TM

Configuration C_1 *yields* configuration C_2 if the Turing machine can legally go from C_1 to C_2 in a single step:

1. uaq_ibv yields uq_jacv if $\delta(q_i, b) = (q_j, c, L)$.
2. uaq_ibv yields $uacq_jv$ if $\delta(q_i, b) = (q_j, c, R)$.
3. q_ibv yields q_jcv if $\delta(q_i, b) = (q_j, c, L)$.
4. q_ibv yields cq_jv if $\delta(q_i, b) = (q_j, c, R)$.

(uaq_i is considered equivalent to $uaq_i\sqcup$.)

Computation of a TM (cont.)

- A Turing machine *accepts* input w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists where
  1. C_1 the start configuration on w,
  2. C_i yields C_{i+1}, and
  3. C_k is an accepting configuration.
- The collection of strings that M accepts is *the language of M*, or *the language recognized by M*, denoted $L(M)$.

Decidable Languages

Definition (3.5)

A language is **Turing-recognizable** (also called *recursively enumerable*) if some Turing machine recognizes it.

- A Turing machine can fail to accept an input by entering the q_{reject} state and rejecting, or by looping (not halting).
- A machine is called a *decider* if it halts on all inputs. A decider that recognizes some language is said to *decide* the language.

Definition (3.6)

A language is **Turing-decidable**, or simply **decidable** (also called *recursive*), if some Turing machine decides it.

Example Turing Machines

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$A = \{0^{2^n} \mid n \geq 0\}$. A decider M_2 for A can be defined to work as follows:

1. Sweep left to right across the tape, crossing off every second 0.
2. If in stage 1 the tape contained a single 0, *accept*.
3. If in stage 1 the tape contained more than one 0 and the number of 0s was odd, *reject*.
4. Return head to the left-hand end of the tape.
5. Go to stage 1.

Example Turing Machines (cont.)

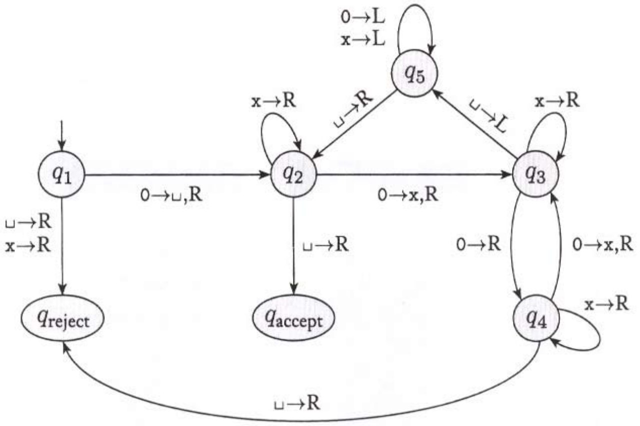

FIGURE 3.8
State diagram for Turing machine M_2

Source: [Sipser 2006]

Example Turing Machines (cont.)

$B = \{w\#w \mid w \in \{0,1\}^*\}$. A decider M_1 for B can be defined to work as follows:

1. Scan the input to be sure that it contains a single $\#$ symbol. If not, *reject*.
2. Zig-zag across the tape to corresponding positions on either side of the $\#$ symbol to check whether these positions contain the same symbol. If they do not, *reject*.
   Cross off symbols as they are checked.
3. When all symbols to the left of the $\#$ have been crossed off, check for any remaining symbols to the right of the $\#$. If any symbols remain, *reject*; otherwise, *accept*.

Example Turing Machines (cont.)

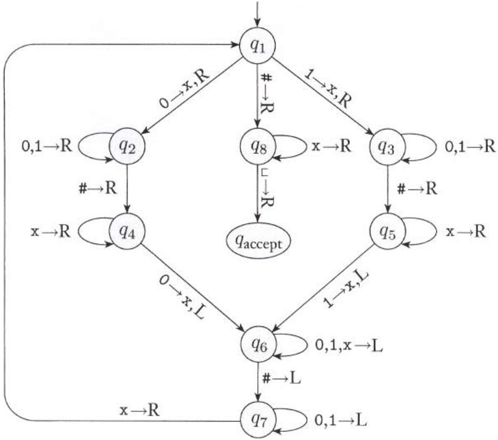

FIGURE 3.10
State diagram for Turing machine M_1

Source: [Sipser 2006]

Example Turing Machines (cont.)

$C = \{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$. A decider M_3 for C:

1. Scan the input to be sure that it is a member of $aa^*bb^*cc^*$ and *reject* if it isn't.
2. Return the head to the left-hand end of the tape.
3. Cross off an a and scan to the right until a b occurs. Shuttle between the b's and c's, crossing off one of each until all b's are gone.
4. Restore the crossed off b's and repeat Stage 3 if there is another a to cross off.
5. If all a's and c's are crossed off, *accept*; otherwise, *reject*.

Example Turing Machines (cont.)

$E = \{\#x_1\#x_2\#\cdots\#x_l \mid x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ (for} i \neq j)\}$.

1. Place a mark on top of the leftmost tape symbol. If that symbol was not a #, *reject*.
2. Scan right to the next # and place a second mark on top of it. If no # occurs before a blank, *accept*.
3. Compare, by zig-zagging, the two strings to the right of the marked #'s. If they are equal, *reject*.
4. Move the second mark to the next # symbol. If not doable, move the first mark to the next # to its right and the second mark to the # after that. If not doable, *accept*.
5. Go to Stage 3.

Variants of Turing Machines

- Alternative definitions of Turing machines abound, including versions with *multiple tapes* or with *nondeterminism*. They are called *variants* of the Turing machine model.
- The original model and its reasonable variants all have the same power—*they recognize the same class of languages*.
- To show that two models are equivalent, we simply need to show that we can *simulate* one by the other.

Multitape Turing Machines

- A *multitape Turing machine* is like an ordinary Turing machine with several tapes.
- Each tape has its own head for reading and writing. Initially the input appears on tape 1 and the others start out blank.
- The transition function is changed to allow for reading, writing, and moving the heads on all the tapes simultaneously. Formally,

$$\delta : Q \times \Gamma^k \longrightarrow Q \times \Gamma^k \times \{L, R, S\}^k,$$

where k is the number of tapes.

Multitape Turing Machines (cont.)

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Theorem (3.13)

Every multitape Turing machine has an equivalent single-tape Turing machine.

A single tape TM S can simulate a k-tape M:

1. S “formats” its tape to represent all k tapes of M:

$$\# \dot{w}_1 \, w_2 \cdots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \cdots \#$$

2. To simulate a single move of M, S scans its tape to determine the symbols under the virtual heads. Then S makes a second pass to update the tapes according to M's transition function.
3. Whenever a virtual head is moved to the right onto a $\#$, S writes a blank symbol on this tape cell and shifts the tape contents from this cell one unit to the right.

Multitape Turing Machines (cont.)

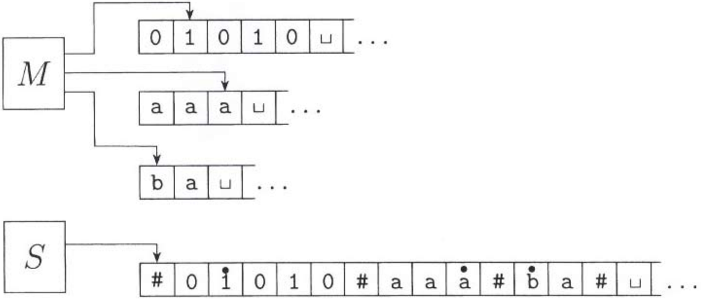

FIGURE 3.14
Representing three tapes with one

Source: [Sipser 2006]

Nondeterministic Turing Machines

- A **nondeterministic Turing machine** is defined in the expected way.
- The transition function of a nondeterministic TM has the form

$$\delta : Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\}).$$

- The computation of a nondeterministic TM is a tree whose branches correspond to different possibilities for the machine.
- If some branch of the computation leads to the accept state, the machine accepts its input.

Nondeterministic Turing Machines (cont.)

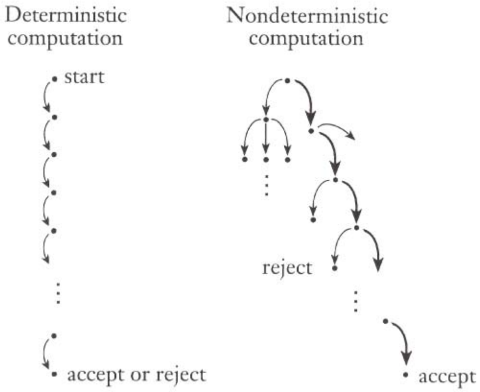

FIGURE 1.28
Deterministic and nondeterministic computations with an accepting branch

Source: [Sipser 2006]

Nondeterministic Turing Machines (cont.)

Theorem (3.16)

Every nondeterministic TM has an equivalent deterministic TM.

- The idea is to have a deterministic TM D try all possible branches of the given nondeterministic TM N's computation.
- D searches, in a breadth first manner, N's computation tree for an accepting configuration.

Nondeterministic Turing Machines (cont.)

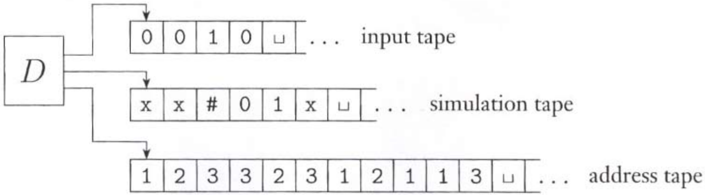

FIGURE 3.17
Deterministic TM D simulating nondeterministic TM N

Source: [Sipser 2006]

Nondeterministic Turing Machines (cont.)

D has three tapes:

- Tape 1 always contains the input string and is never altered.
- Tape 2 maintains a copy of *N*'s tape on some branch of its nondeterministic computation.
- Tape 3 keeps track of *D*'s location in *N*'s nondeterministic computation tree.

Enumerators

- Some people use the term **recursively enumerable** language for Turing-recognizable language.
- An *enumerator* is a Turing machine with an attached printer. Every time the Turing machine wants to add a string to the output list, it sends the string to the printer.
- The language enumerated by an enumerator E is the collection of all the strings that E eventually prints out.
- Moreover, E may generate the strings of the language in any order, possibly with repetitions.

Enumerators (cont.)

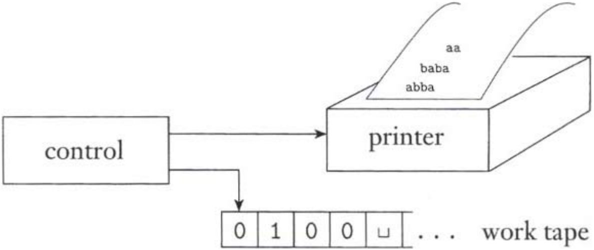

FIGURE 3.20
Schematic of an enumerator

Source: [Sipser 2006]

Enumerators (cont.)

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Theorem (3.21)

A language is Turing-recognizable if and only if some enumerator enumerates it.

To recognize the language enumerated by E, a TM M works as follows:

1. Run E. Every time that E outputs a string, compare it with the input w.
2. If w appears in the output of E, *accept*.

Enumerators (cont.)

To enumerate the language recognized by M, an enumerator E works as follows:

1. Repeat Steps 2 and 3 for $i = 1, 2, 3, \ldots$
2. Run M for i steps on each input, s_1, s_2, ..., s_i.
3. If any computations accept, print out the corresponding s_j.

Hilbert's Tenth Problem

- A *polynomial* is a sum of terms, where each term is a product of variables and a constant.
- For example, $6x^3yz^2 + 3xy^2 - x^3 - 10$ is a polynomial with four terms over variables x, y, and z.
- Let $D = \{p \mid p \text{ is a polynomial with an integral root}\}$.
- Hilbert's tenth problem (rephrased): "Is there an algorithm for determining D?"
- Proving that no algorithm exists for a particular task requires a precise definition of algorithm.

Hilbert's Tenth Problem: The Original Statement

10. Determination of the solvability of a Diophantine equation. Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

Note: The kind of process that Hilbert looked after is "effective procedure" and is nowadays referred to as "computer algorithm" or simply "algorithm."

“Effective” Procedures

A procedure M is considered effective if the following hold:

1. M contains a finite number of exact instructions (each being expressed with a finite number of symbols);
2. M will, if carried out without error, always produce the desired result in a finite number of steps;
3. M can (in practice or in principle) be carried out by a human being unaided by any machinery save paper and pencil;
4. M demands no insight or ingenuity on the part of the human being carrying it out.

Note: excerpted from “The Church-Turing Thesis” of *Stanford Encyclopedia of Philosophy*.

The Definition of Algorithm

- All models of a general-purpose computer turn out to be at best equivalent in power to the Turing machine, as long as they satisfy certain reasonable requirements.
- This has an important philosophical corollary: Even though there are many different computational models, *the class of algorithms that they describe is unique*.
- The **Church-Turing thesis** says that *the intuitive notion of an algorithm corresponds to the formal definition of a Turing machine*.

Describing Turing Machines

Three possible levels of detail:

- A *formal description* spells out in full the Turing machine's states, transition function, and so on.
- In an *implementation description*, we use natural language prose to describe the way that the Turing machine moves its head and the way that it stores data on its tape.
- In a *high-level description*, we use natural language prose to describe an algorithm, ignoring the implementation model.

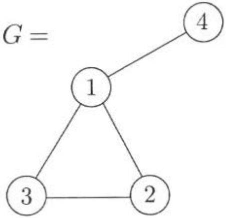
An Example High-Level Description

Let $A = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$. The following is a high-level description of a TM M that decides A:

$M =$ "On input $\langle G \rangle$, the encoding of a graph G:

1. Select the first node of G and mark it.
2. Repeat Step 3 until no new nodes are marked.
3. For each node in G, mark it if it is attached by an edge to a node that is already marked.
4. Scan all the nodes of G to determine whether they all are marked. If they are, *accept*; otherwise, *reject*."

An Example High-Level Description (cont.)



$\langle G \rangle =$

$(1,2,3,4)((1,2),(2,3),(3,1),(1,4))$

FIGURE 3.24
A graph G and its encoding $\langle G \rangle$

Source: [Sipser 2006]