Minimization of DFAs

(Based on [Sipser 2013] and [Wikipedia])

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Distinguishable and Indistinguishable Strings

- Let L be a language over Σ (i.e., $L \subseteq \Sigma^*$).
- Two strings x and y in Σ^* are **distinguishable by L** if
  for some string z in Σ^*, exactly one of xz and yz is in L.
- When no such z exists, i.e.,
  for every z in Σ^*, either both of xz and yz or neither of them are in L,
  we say that x and y are **indistinguishable by L**.
- Indistinguishable strings can be regarded as essentially equivalent.

Note: these concepts apply to all languages, not just the regular ones.

Myhill-Nerode Theorem

- Given a language $L \subseteq \Sigma^*$, define a binary relation R_L over Σ^* as follows:

$$xR_Ly \quad \text{iff} \quad \forall z \in \Sigma^*(xz \in L \leftrightarrow yz \in L)$$

- R_L can be shown to be an equivalence relation.

Theorem (Myhill-Nerode)

With R_L defined as above, the following are equivalent:

1. *L is regular.*
2. *R_L is of finite index.*

Moreover, the index of R_L equals the number of states in the smallest DFA that recognizes L.

Note: the *index* of an equivalence relation is the number of equivalence classes it induces.

Minimization of DFAs

- A DFA $(Q, \Sigma, \delta, q_0, F)$ for L defines an equivalence relation on Σ^* that is a *refinement* of R_L.
- Let $L_q = \{x \in \Sigma^* \mid \delta(q_0, x) = q\}$. Then,
  - for distinct $q, q' \in Q$, $L_q \cap L_{q'} = \emptyset$, and
  - for every $q \in Q$, L_q is contained in an equivalence class of R_L.
- Given a DFA that is not minimum for its language, there must be two distinct states q and q' such that both L_q and $L_{q'}$ are contained in the same equivalence class of R_L and hence may be merged (without affecting the language recognized).

Minimization of DFAs (cont.)

- On the opposite, there are states that must remain separate.
- Apparently, for $q \in F$ and $q' \in Q \setminus F$, L_q and $L_{q'}$ are in different equivalence classes of R_L and hence q and q' must remain separate.
- For any two states, if they can make a transition on the same symbol to two different states that should remain separate, then they should also remain separate; this should be checked repeatedly.
- To minimize a DFA, we may start with the partition $\{F, Q \setminus F\}$ and refine the partition by iteratively checking whether two states in the same block should be separated.

Hopcroft's Minimization Algorithm

Algorithm Minimization(Q, Σ, δ, F);
begin
    $P := \{F, Q \setminus F\}$; $W := \{F\}$;
    while W not empty **do**
        remove a set A from W;
        for each $c \in \Sigma$ **do**
            $X := \{q \mid \delta(q, c) \in A\}$;
            for each $Y \in P$ s.t. both $X \cap Y$ and $Y \setminus X$ not empty **do**
                replace Y in P by $X \cap Y$ and $Y \setminus X$;
                if $Y \in W$ **then**
                    replace Y in W by $X \cap Y$ and $Y \setminus X$
                else if $|X \cap Y| \leq |Y \setminus X|$ **then**
                        add $X \cap Y$ to W
                    else add $Y \setminus X$ to W
end