Theory of Computing

Introduction and Preliminaries
(Based on [Sipser 2006, 2013])

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What It Is

🌐 The central question:

What are the fundamental capabilities and limitations of computers?

🌐 Three main areas:
  - ☀ *Automata Theory*
  - ☀ *Computability Theory*
  - ☀ *Complexity Theory*

Complexity Theory

- Some problems are easy and some hard.
  For example, sorting is easy and scheduling is hard.

- The central question of complexity theory:
  What makes some problems computationally hard and others easy?

- We don't have the answer to it.

- However, researchers have found a scheme for classifying problems according to their computational difficulty.

- One practical application: cryptography/security.

Dealing with Computationally Hard Problems

Options for dealing with a computationally hard problem:

- 🔵 Try to simplify it (the hard part of the problem might be unnecessary).
- 🔵 Settle for an approximate solution.
- 🔵 Find a solution that usually runs fast.
- 🔵 Consider alternative types of computation (such as randomized computation).

Computability Theory

- Alan Turing, among other mathematicians, discovered in the 1930s that certain basic problems cannot be solved by computers.

- One example is the problem of determining whether a mathematical statement is true or false.

- Theoretical models of computers developed at that time eventually lead to the construction of actual computers.

- The theories of computability and complexity are closely related.

- *Complexity theory* seeks to classify problems as easy ones and hard ones, while in *computability theory* the classification is by whether the problem is solvable or not.

Automata Theory

- The theories of computability and complexity require a precise, formal definition of a *computer*.
- *Automata theory* deals with the definitions and properties of mathematical models of computation.
- Two basic and practically useful models:
  - *Finite-state*, or simply *finite*, *automaton*
  - *Context-free grammar* (pushdown automaton)

Why You Should Learn the Subject

🌐 It will certainly broaden your knowledge of what computing is fundamentally.

🌐 Below are a few things you may find particularly useful or interesting:

☀ Regular expressions, in their original simplest form, for describing patterns of strings/words.

☀ Context-free grammars for describing the syntax of a (programming) language.

☀ The so-called Turing machines, as the most commonly used model for a computer.

☀ Exemplar undecidable problems, which cannot be (perfectly) solved by computers.

☀ A proof of SAT being NP-hard, where every NP problem is shown to be polynomially reducible to SAT.

Sets

- Set, element (member), subset, proper subset
- Multiset
- Description of a set
- The empty set ($\emptyset$)
- Finite set, infinite set
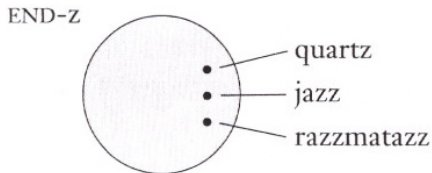- Union, intersection, complement
- Power set
- Venn diagram

Sets (cont.)

START-t
- terrific
- tundra
- theory

FIGURE 0.1
Venn diagram for the set of English words starting with "t"

Source: [Sipser 2006]

Sets (cont.)

END-Z
- quartz
- jazz
- razzmatazz

FIGURE **0.2**
Venn diagram for the set of English words ending with "z"
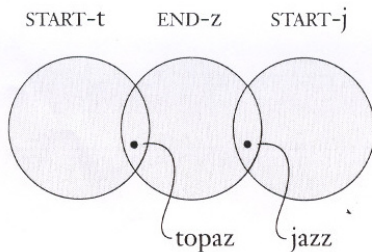
Source: [Sipser 2006]

Sets (cont.)

FIGURE **0.3**
Overlapping circles indicate common elements
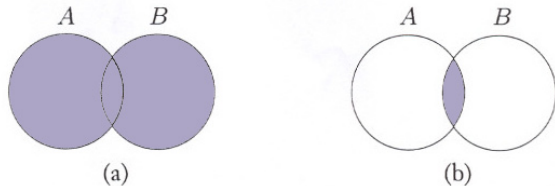
Source: [Sipser 2006]

FIGURE **0.4**
Diagrams for (a) $A \cup B$ and (b) $A \cap B$

Source: [Sipser 2006]

Sequences and Tuples

- A *sequence* of objects is a list of these objects in some order. Order is essential and repetition is also allowed.

- Finite sequences are often called *tuples*. A sequence with k elements is a *k-tuple*; a 2-tuple is also called a *pair*.

- The *Cartesian product*, or cross product, of A and B, written as $A \times B$, is the set of all pairs (x, y) such that $x \in A$ and $y \in B$.

- Cartesian products generalize to k sets, A_1, A_2, ..., A_k, written as $A_1 \times A_2 \times \ldots \times A_k$. Every element in the product is a k-tuple.

- A^k is a shorthand for $A \times A \times \ldots \times A$ (k times).

Functions

- A *function* sets up an *input-output* relationship, where the same input always produces the same output.
- If f is a function whose output is b when the input is a, we write $f(a) = b$.
- A function is also called a *mapping*; if $f(a) = b$, we say that f maps a to b.

Functions (cont.)

- The set of possible inputs to a function is called its *domain*; the outputs come from a set called its *range*.
- A function is *onto* if it uses all the elements of the range (it is *one-to-one* if . . .).
- The notation $f : D \longrightarrow R$ says that f is a function with domain D and range R.
- More notions and terms: *k-ary function*, *unary function*, *binary function*, *infix notation*, *prefix notation*

Relations

- A *predicate*, or property, is a function whose range is {TRUE,FALSE}.
- A predicate whose domain is $A_1 \times A_2 \times \ldots \times A_k$ is called a *k*-ary *relation* on $A_1, A_2, \ldots, A_k$. When the A_i's are the same set A, it is simply called a *k*-ary relation on A.
- A 1-ary relation is usually called a *unary relation* and a 2-ary relation is called a *binary relation*.

Equivalence Relations

- An *equivalence relation* is a special type of binary relation that captures the notion of two objects being *equal* in some sense.
- A binary relation R on A is an equivalence relation if
  1. R is *reflexive* (for every x in A, xRx),
  2. R is *symmetric* (for every x and y in A, xRy if and only if yRx), and
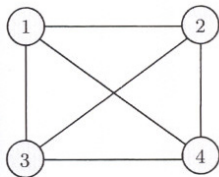 3. R is *transitive* (for every x, y, and z in A, xRy and yRz implies xRz).

Graphs

- Undirected graph, node (vertex), edge (link), degree
- Description of a graph: $G = (V, E)$
- Labeled graph
- Subgraph, induced subgraph
- Path, simple path, cycle, simple cycle
- Connected graph
- Tree, root, leaf
- Directed graph, outdegree, indegree
- Strongly connected graph

Graphs (cont.)
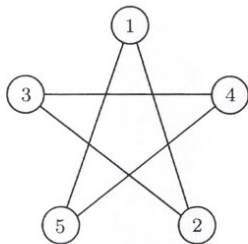
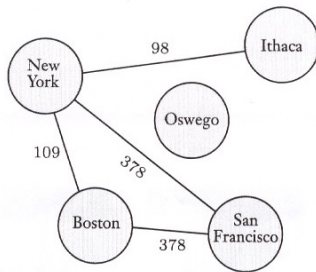

FIGURE **0.12**
Examples of graphs

Source: [Sipser 2006]

Graphs (cont.)

FIGURE **0.13**
Cheapest nonstop air fares between various cities

Source: [Sipser 2006]

Graphs (cont.)
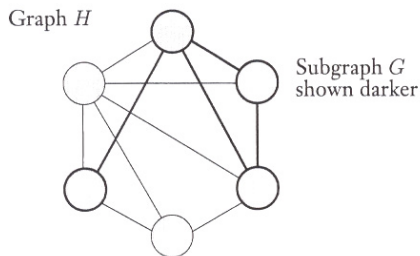

Graph H

Subgraph G
shown darker

FIGURE **0.14**
Graph G (shown darker) is a subgraph of H

Source: [Sipser 2006]
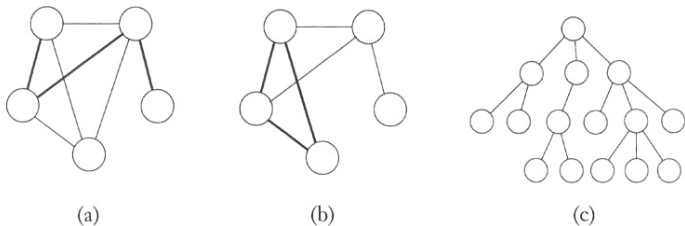
Graphs (cont.)

FIGURE **0.15**
(a) A path in a graph, (b) a cycle in a graph, and (c) a tree
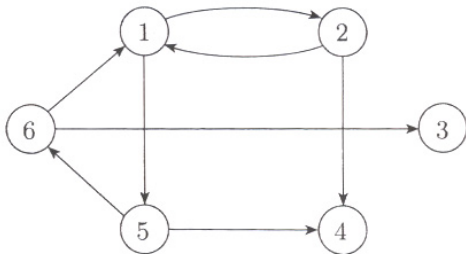
Source: [Sipser 2006]

Graphs (cont.)

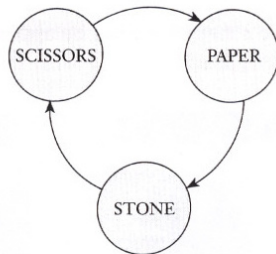FIGURE **0.16**
A directed graph

Source: [Sipser 2006]

FIGURE **0.18**
The graph of the relation *beats*

Source: [Sipser 2006]

Strings and Languages

- An *alphabet* is any finite set of *symbols*.
- A *string* over an alphabet is a finite sequence of symbols from that alphabet.
- The *length* of a string w, written as $|w|$, is the number of symbols that w contains.
- The string of length 0 is called the *empty string*, written as ε.
- The *concatenation* of x and y, written as xy, is the string obtained from appending y to the end of x.
- A *language* is a set of strings.
- More notions and terms: *reverse*, *substring*, *lexicographic ordering*.

Boolean Logic

🌑 *Boolean logic* is a mathematical system built around the two *Boolean values* TRUE (1) and FALSE (0).

🌑 Boolean values can be manipulated with *Boolean operations*: *negation* or NOT ($\neg$), *conjunction* or AND ($\wedge$), *disjunction* or OR ($\vee$).

$$
\begin{array}{lll}
0 \wedge 0 \triangleq 0 & 0 \vee 0 \triangleq 0 & \neg 0 \triangleq 1 \\
0 \wedge 1 \triangleq 0 & 0 \vee 1 \triangleq 1 & \neg 1 \triangleq 0 \\
1 \wedge 0 \triangleq 0 & 1 \vee 0 \triangleq 1 & \\
1 \wedge 1 \triangleq 1 & 1 \vee 1 \triangleq 1 &
\end{array}
$$

🌑 Unknown Boolean values are represented symbolically by *Boolean variables* or *propositions*, e.g., P, Q, etc.

Boolean Logic (cont.)

🔵 Additional Boolean operations: *exclusive or* or XOR ($\oplus$), *equality/equivalence* ($\leftrightarrow$ or $\equiv$), *implication* ($\rightarrow$).

$$0 \oplus 0 \stackrel{\Delta}{=} 0 \qquad 0 \leftrightarrow 0 \stackrel{\Delta}{=} 1 \qquad 0 \rightarrow 0 \stackrel{\Delta}{=} 1$$
$$0 \oplus 1 \stackrel{\Delta}{=} 1 \qquad 0 \leftrightarrow 1 \stackrel{\Delta}{=} 0 \qquad 0 \rightarrow 1 \stackrel{\Delta}{=} 1$$
$$1 \oplus 0 \stackrel{\Delta}{=} 1 \qquad 1 \leftrightarrow 0 \stackrel{\Delta}{=} 0 \qquad 1 \rightarrow 0 \stackrel{\Delta}{=} 0$$
$$1 \oplus 1 \stackrel{\Delta}{=} 0 \qquad 1 \leftrightarrow 1 \stackrel{\Delta}{=} 1 \qquad 1 \rightarrow 1 \stackrel{\Delta}{=} 1$$

🔵 All in terms of conjunction and negation:

$$
\begin{aligned}
P \vee Q &\equiv \neg(\neg P \wedge \neg Q) \\
P \rightarrow Q &\equiv \neg P \vee Q \\
P \leftrightarrow Q &\equiv (P \rightarrow Q) \wedge (Q \rightarrow P) \\
P \oplus Q &\equiv \neg(P \leftrightarrow Q)
\end{aligned}
$$

Logical Equivalences and Laws

- Two logical expressions/formulae are *equivalent* if each of them implies the other, i.e., they have the same truth value.
- Equivalence plays a role analogous to equality in algebra.
- Some laws of Boolean logic:
    - (Distributive) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
    - (Distributive) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
    - (De Morgan's) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
    - (De Morgan's) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Definitions, Theorems, and Proofs

- *Definitions* describe the objects and notions that we use. Precision is essential to any definition.

- After we have defined various objects and notions, we usually make *mathematical statements* about them. Again, the statements must be precise.

- A *proof* is a convincing logical argument that a statement is true. The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof.

- A *theorem* is a mathematical statement proven true. *Lemmas* are proven statements for assisting the proof of another more significant statement.

- *Corollaries* are statements seen to follow easily from other proven ones.

Finding Proofs

🌐 Find proofs isn't always easy; no one has a recipe for it.

🌐 Below are some helpful general strategies:

  1. Carefully read the statement you want to prove.
  2. Rewrite the statement in your own words.
  3. Break it down and consider each part separately.
     For example, $P \iff Q$ consists of two parts: $P \to Q$ (the forward direction) and $Q \to P$ (the reverse direction).
  4. Try to get an intuitive feeling of why it should be true.

Tips for Producing a Proof

- A well-written proof is a sequence of statements, wherein each one follows by simple reasoning from previous statements in the sequence.

- Tips for producing a proof:
    - *Be patient*. Finding proofs takes time.
    - *Come back to it*. Look over the statement, think about it, leave it, and then return some time later.
    - *Be neat*. Use simple, clear text and/or pictures; make it easy for others to understand.
    - *Be concise*. Emphasize high-level ideas, but be sure to include enough details of reasoning.

An Example Proof

Theorem

For any two sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. We show that every element of $\overline{A \cup B}$ is also an element of $\overline{A} \cap \overline{B}$ and vice versa.

Forward ($x \in \overline{A \cup B} \rightarrow x \in \overline{A} \cap \overline{B}$):

$\quad\quad x \in \overline{A \cup B}$

$\rightarrow \quad x \notin A \cup B$, def. of complement

$\rightarrow \quad x \notin A$ and $x \notin B$, def. of union

$\rightarrow \quad x \in \overline{A}$ and $x \in \overline{B}$, def. of complement

$\rightarrow \quad x \in \overline{A} \cap \overline{B}$, def. of intersection

Reverse ($x \in \overline{A} \cap \overline{B} \rightarrow x \in \overline{A \cup B}$): ...
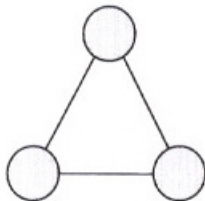
Another Example Proof

Theorem

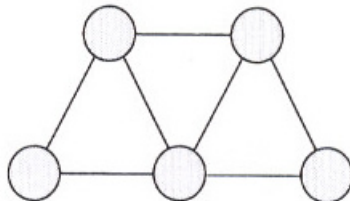*In any graph G, the sum of the degrees of the nodes of G is an even number.*

Proof.

- 🔵 Every edge in G connects two nodes, contributing 1 to the degree of each.
- 🔵 Therefore, each edge contributes 2 to the sum of the degrees of all the nodes.
- 🔵 If G has e edges, then the sum of the degrees of the nodes of G is $2e$, which is even.
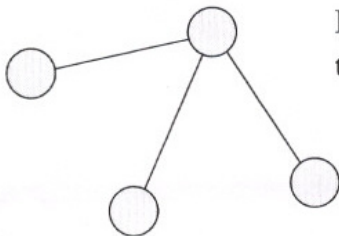
Another Example Proof (cont.)

$$\text{sum} = 2+2+2$$
$$= 6$$

$$\text{sum} = 2+3+4+3+2$$
$$= 14$$

Source: [Sipser 2006]

Another Example Proof (cont.)

Every time an edge is added, the sum increases by 2.

Source: [Sipser 2006]

Types of Proof

- *Proof by construction*:
  prove that a particular type of object exists, by showing how to construct the object.

- *Proof by contradiction*:
  prove a statement by first assuming that the statement is false and then showing that the assumption leads to an obviously false consequence, called a contradiction.

- *Proof by induction*:
  prove that all elements of an infinite set have a specified property, by exploiting the inductive structure of the set.

Proof by Construction

Theorem

For each even number n greater than 2*, there exists a* 3*-regular graph with n nodes.*

Proof. Construct a graph $G = (V, E)$ with $n\ (= 2k > 2)$ nodes as follows.

Let V be $\{0, 1, \ldots, n - 1\}$ and E be defined as

$$E = \{\{i, i + 1\} \mid \text{for } 0 \le i \le n - 2\} \cup$$
$$\{\{n - 1, 0\}\} \cup$$
$$\{\{i, i + n/2\} \mid \text{for } 0 \le i \le n/2 - 1\}.$$

Proof by Contradiction

Theorem

$\sqrt{2}$ is irrational.

Proof. Assume toward a contradiction that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = \frac{m}{n}$ for some integers m and n, which *cannot both be even*.

$\sqrt{2} = \frac{m}{n}$	, from the assumption
$n\sqrt{2} = m$	, multipl. both sides by n
$2n^2 = m^2$	, square both sides
m is even	, m^2 is even
$2n^2 = (2k)^2 = 4k^2$	, from the above two
$n^2 = 2k^2$	, divide both sides by 2
n is even	, n^2 is even

Now both m and n are even, a contradiction.

Example: Home Mortgages

P: the *principle* (amount of the original loan).

I: the yearly *interest rate*.

Y: the monthly payment.

M: the *monthly* multiplier $= 1 + I/12$.

P_t: the amount of loan outstanding after the t-th month; $P_0 = P$ and $P_{k+1} = P_k M - Y$.

Theorem

For each $t \geq 0$,

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right).$$

Proof by Induction

Theorem

For each $t \geq 0$,

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right).$$

Proof. The proof is by induction on t.

- 🌐 *Basis*: When $t = 0$, $PM^0 - Y\left(\frac{M^0 - 1}{M - 1}\right) = P = P_0$.

Proof by Induction (cont.)

🌐 *Induction step*: When $t = k + 1$ ($k \geq 0$),

$$
\begin{aligned}
& P_{k+1} \\
= \quad & \{\text{definition of } P_t\} \\
& P_k M - Y \\
= \quad & \{\text{the induction hypothesis}\} \\
& (PM^k - Y(\tfrac{M^k - 1}{M - 1}))M - Y \\
= \quad & \{\text{distribute } M \text{ and rewrite } Y\} \\
& PM^{k+1} - Y(\tfrac{M^{k+1} - M}{M - 1}) - Y(\tfrac{M - 1}{M - 1}) \\
= \quad & \{\text{combine the last two terms}\} \\
& PM^{k+1} - Y(\tfrac{M^{k+1} - 1}{M - 1})
\end{aligned}
$$

Structural Induction

- Structural induction is a generalization of mathematical induction on the natural numbers.

- It is used to prove that some proposition $P(x)$ holds for all x of some sort of recursively/inductively defined structure such as binary trees.

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- Proof by structural induction:
    1. Base case: the proposition holds for all the minimal structures.
    2. Inductive step: if the proposition holds for the immediate substructures of a certain structure S, then it also holds for S.