Theory of Computing 2022: Reducibility

(Based on [Sipser 2006, 2013])

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1 Introduction

Introduction

- A *reduction* is a way of converting one problem into another problem in such a way that a solution to the second problem can be used to solve the first problem.
- If a problem A reduces (is reducible) to another problem B, we can use a solution to B to solve A.
- *Reducibility* says nothing about solving A or B alone, but only about the solvability of A in the presence of a solution to B.
- Reducibility is the primary method for proving that problems are computationally unsolvable.
- Suppose that A is reducible to B. If B is decidable, then A is decidable; equivalently, if A is undecidable, then B is undecidable.

2 Undecidable Problems

The Halting Problem

- $HALT_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on } w\}$.

Theorem 1 (5.1)**.** *$HALT_{\text{TM}}$ is undecidable.*

- The idea is to reduce the acceptance problem A_{TM} (shown to be undecidable) to $HALT_{\text{TM}}$.
- Assume toward a contradiction that a TM R decides $HALT_{\text{TM}}$.
- We could then construct a decider S for A_{TM} as follows.

The Halting Problem (cont.)

S = "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Run TM R on input $\langle M, w\rangle$.
2. If R rejects, reject.
3. If R accepts, simulate M on w until it halts.
4. If M has accepted, *accept*; if M has rejected, reject."

Undecidable Problems

- $E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$.

Theorem 2 (5.2)**.** *E_{TM} is undecidable.*

- Assuming that a TM R decides E_{TM}, we construct a decider S for A_{TM} as follows.

Undecidable Problems (cont.)

$S =$ "On input $\langle M, w \rangle$:

1. Construct the following TM M_1.

   $M_1 =$ "On input x:

   (a) If $x \neq w$, reject.

   (b) If $x = w$, run M on input w and *accept* if M accepts w."

2. Run R on input $\langle M_1 \rangle$.

3. If R accepts, reject; if R rejects, *accept*."

Undecidable Problems (cont.)

- $REGULAR_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$.

Theorem 3 (5.3)**.** *$REGULAR_{\text{TM}}$ is undecidable.*

- Assuming that a TM R decides $REGULAR_{\text{TM}}$, we construct a decider S for A_{TM} as follows.

Undecidable Problems (cont.)

$S =$ "On input $\langle M, w \rangle$:

1. Construct the following TM M_2.

   $M_2 =$ "On input x:

   (a) If x has the form $0^n 1^n$, *accept*.

   (b) If x does not have this form, run M on input w and *accept* if M accepts w."

2. Run R on input $\langle M_2 \rangle$.

3. If R accepts, *accept*; if R rejects, reject."

Note: if M does not accept w, then $L(M_2) = \{0^n 1^n \mid n \geq 0\}$, which is not regular; if M accepts w, then $L(M_2) = \{0, 1\}^*$, which is regular.

Undecidable Problems (cont.)

- $EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$.

Theorem 4 (5.4)**.** *EQ_{TM} is undecidable.*

- Assume that a TM R decides EQ_{TM}.
- We construct a decider S for E_{TM} as follows.
- $S =$ "On input $\langle M \rangle$:
  1. Run R on input $\langle M, M_1 \rangle$, where M_1 is a TM that rejects all inputs.
  2. If R accepts, *accept*; if R rejects, reject."

Rice's Theorem

Theorem 5. *Any "nontrivial" property about the languages recognized by Turing machines is undecidable.*

- Note 1: the theorem considers only properties about languages, i.e., properties that do not distinguish equivalent Turing machine descriptions.
- Note 2: a property is *nontrivial* if it is satisfied by some, but not all, Turing machine descriptions.

3 Reduction via Computation Histories

Computation Histories

Definition 6 (5.5)**.** An *accepting computation history* for M on w is a sequence of configurations $C_1, C_2, \cdots, C_l$, where

1. C_1 is the start configuration,
2. C_l is an accepting configuration, and
3. C_i yields C_{i+1}, $1 \leq i \leq l-1$.

A *rejecting computation history* for M on w is defined similarly, except that C_l is a rejecting configuration.

- Computation histories are finite sequences.
- Deterministic machines have at most one computation history on any given input.

Linear Bounded Automata

Definition 7 (5.6)**.** A *linear bounded automaton* (LBA) is a restricted type of Turing machine wherein the tape head is not permitted to move off the portion of the tape containing the input.

- So, an LBA is a TM with a limited amount of memory. It can only solve problems requiring memory that can fit within the tape used for the input.

(Note: using a tape alphabet larger than the input alphabet allows the available memory to be increased up to a constant factor.)

Linear Bounded Automata (cont.)

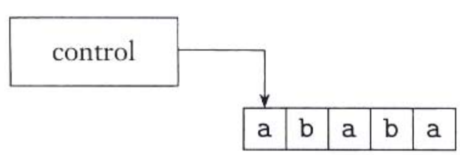


FIGURE 5.7
Schematic of a linear bounded automaton

Source: [Sipser 2006]

Linear Bounded Automata (cont.)

Despite their memory constraint, LBAs are quite powerful.

Lemma 8 (5.8)**.** *Let M be an LBA with q states and g symbols in the tape alphabet. There are exactly qng^n distinct configurations of M for a tape of length n.*

Decidable Problems about LBAs

- $A_{\text{LBA}} = \{\langle M, w\rangle \mid M$ is an LBA that accepts $w\}$.

Theorem 9 (5.9)**.** A_{LBA} *is decidable.*

- $L =$ "On input $\langle M, w\rangle$, an encoding of an LBA M and a string w:
  1. Simulate M on input w for qng^n steps or until it halts.
  2. If M has halted, *accept* if it has accepted and reject if it has rejected. If M has not halted, reject."

Undecidable Problems about LBAs

- $E_{\text{LBA}} = \{\langle M\rangle \mid M$ is an LBA where $L(M) = \emptyset\}$.

Theorem 10 (5.10)**.** E_{LBA} *is undecidable.*

- Assuming that a TM R decides E_{LBA}, we construct a decider S for A_{TM} as follows.
- $S =$ "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:
  1. Construct an LBA B from $\langle M, w\rangle$ that, on input x, decides whether x is an accepting computation history for M on w.
  2. Run R on input $\langle B\rangle$.
  3. If R rejects, *accept*; if R accepts, reject."

Undecidable Problems about LBAs (cont.)

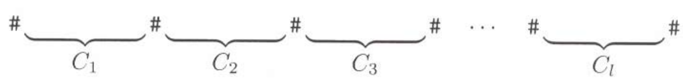


FIGURE 5.11
A possible input to B

Source: [Sipser 2006]

Three conditions of an accepting computation history:

- C_1 is the start configuration.
- C_l is an accepting configuration.
- C_i yields C_{i+1}, for every i, $1 \leq i < l$.

Undecidable Problems about LBAs (cont.)

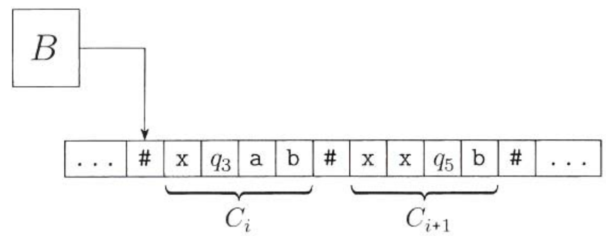


FIGURE 5.12
LBA B checking a TM computation history

Source: [Sipser 2006]

Undecidable Problems about CFGs

- $ALL_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$.

Theorem 11 (5.13). *ALL_{CFG} is undecidable.*

- For a TM M and an input w, we construct a CFG G (by first constructing a PDA) to generate all strings that are *not* accepting computation histories for M on w.
- That is, G generates all strings if and only if M does not accept w.
- If ALL_{CFG} were decidable, then A_{TM} would be decidable.

Undecidable Problems about CFGs (cont.)

The PDA for recognizing computation histories that are not accepting works as follows.

- The input is regarded as a computation history of the form:

$$\#C_1\#C_2^R\#C_3\#C_4^R\#\cdots\#C_l\#$$

  where C_i^R denotes the reverse of C_i.

- The PDA nondeterministically chooses to check if one of the following conditions holds for the input:
  - C_1 is not the start configuration.
  - C_l is not an accepting configuration.
  - C_i does not yield C_{i+1}, for some i, $1 \leq i < l$.
- It also accepts an input that is not in the proper form of a computation history.

Undecidable Problems about CFGs (cont.)

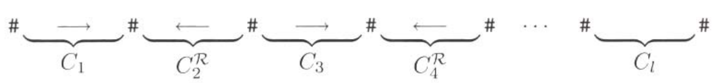


FIGURE 5.14
Every other configuration written in reverse order

Source: [Sipser 2006]

4 The Post Correspondence Problem

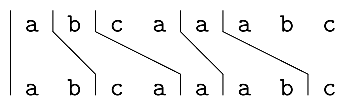
The Post Correspondence Problem

- Consider a collection of dominos such as follows:

$$\left\{ \left[\frac{b}{ca}\right], \left[\frac{a}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right] \right\}$$

- A *match* is a list of these dominos (repetitions permitted) where the string of symbols on the top is the same as that on the bottom. Below is a match:

$$\left[\frac{a}{ab}\right] \left[\frac{b}{ca}\right] \left[\frac{ca}{a}\right] \left[\frac{a}{ab}\right] \left[\frac{abc}{c}\right]$$



The Post Correspondence Problem (cont.)

- The Post correspondence problem (PCP) is to determine whether a collection of dominos has a match.
- More formally, an instance of the PCP is a collection of dominos:

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \cdots, \left[\frac{t_k}{b_k}\right] \right\}$$

- A *match* is a sequence $i_1, i_2, \cdots, i_l$ such that $t_{i_1} t_{i_2} \cdots t_{i_l} = b_{i_1} b_{i_2} \cdots b_{i_l}$.
- $PCP = \{\langle P \rangle \mid P$ is an instance of the Post correspondence problem with a match$\}$.

Undecidability of the PCP

Theorem 12 (5.15). *PCP is undecidable*

- The proof is by reduction from A_{TM} via accepting computation histories.
- From any TM M and input w we can construct an instance P where a match is an accepting computation history for M on w.
- Assume that a TM R decides PCP.
- A decider S for A_{TM} constructs an instance of the PCP that has a match if and only if M accepts w, as follows.

Undecidability of the PCP (cont.)

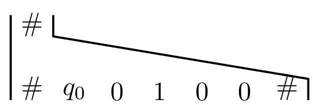
1. Add $\left[\frac{\#}{\#q_0w_1w_2\cdots w_n\#}\right]$ as $\left[\frac{t_1}{b_1}\right]$.
2. For every $a, b \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{\text{reject}}$,

   if $\delta(q, a) = (r, b, R)$, add $\left[\frac{qa}{br}\right]$.
3. For every $a, b, c \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{\text{reject}}$,

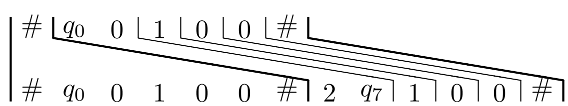
   if $\delta(q, a) = (r, b, L)$, add $\left[\frac{cqa}{rcb}\right]$.
4. For every $a \in \Gamma$, add $\left[\frac{a}{a}\right]$.
5. Add $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$.

Undecidability of the PCP (cont.)

A start configuration (by Part 1):

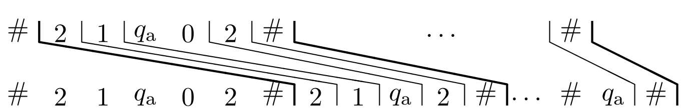


Suppose $\delta(q_0, 0) = (q_7, 2, R)$. With Parts 2-5, the match may be extended to:

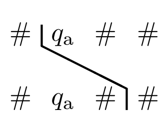


Undecidability of the PCP (cont.)

6. For every $a \in \Gamma$, add $\left[\frac{aq_{\text{accept}}}{q_{\text{accept}}}\right]$ and $\left[\frac{q_{\text{accept}}a}{q_{\text{accept}}}\right]$.



7. Add $\left[\frac{q_{\text{accept}}\#\#}{\#}\right]$.



Undecidability of the PCP (cont.)

To ensure that a match starts with $\left[\frac{t_1}{b_1}\right]$,

S converts the collection $\left\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \cdots, \left[\frac{t_k}{b_k}\right]\right\}$ to

$$\left\{\left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_1}{b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \cdots, \left[\frac{\star t_k}{b_k \star}\right], \left[\frac{* \Diamond}{\Diamond}\right]\right\}$$

where

$$\begin{array}{rcl} \star u & = & * u_1 * u_2 * u_3 * \cdots * u_n \\ u \star & = & u_1 * u_2 * u_3 * \cdots * u_n * \\ \star u \star & = & * u_1 * u_2 * u_3 * \cdots * u_n * \end{array} .$$

5 Mapping Reducibility

Computable Functions

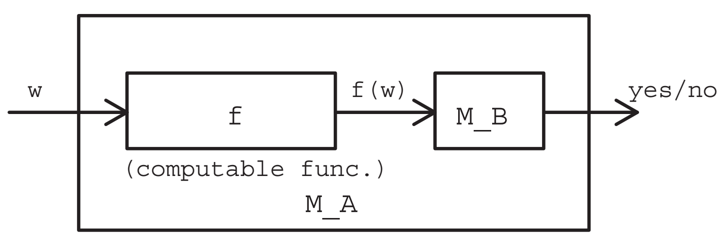
- A Turing machine computes a function by starting with the input to the function on the tape and halting with the output of the function on the tape.

  Definition 13 (5.17)**.** A function $f : \Sigma^* \longrightarrow \Sigma^*$ is a **computable function** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

- For example, all usual arithmetic operations on integers are computable functions.
- Computable functions may be transformations of machine descriptions.

Mapping (Many-One) Reducibility

Definition 14 (5.20)**.** Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w, $w \in A \Longleftrightarrow f(w) \in B$.



- This provides a way to convert questions about membership testing in A to membership testing in B.

Mapping (Many-One) Reducibility (cont.)

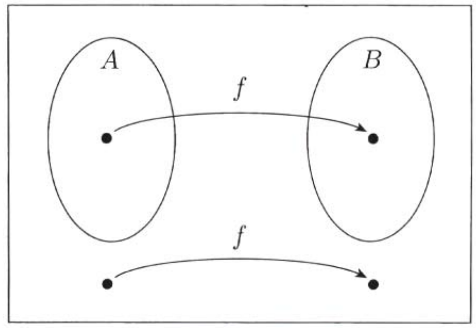


FIGURE 5.21
Function f reducing A to B

Source: [Sipser 2006]

- The function f is called the *reduction* of A to B.

Reducibility and Decidability

Theorem 15 (5.22)**.** *If $A \leq_m B$ and B is decidable, then A is decidable.*

- Let M be a decider for B and f a reduction from A to B. A decider N for A works as follows.
- $N =$ "On input w:
  1. Compute $f(w)$.
  2. Run M on input $f(w)$ and output whatever M outputs."

Corollary 16 (5.23)**.** *If $A \leq_m B$ and A is undecidable, then B is undecidable.*

Note: $(P \wedge Q) \rightarrow R \equiv P \rightarrow (Q \rightarrow R) \equiv P \rightarrow (\neg R \rightarrow \neg Q) \equiv (P \wedge \neg R) \rightarrow \neg Q$

Reducibility and Decidability (cont.)

Theorem 17. *$HALT_{\mathrm{TM}}$ is undecidable.*

- We show that $A_{\mathrm{TM}} \leq_m HALT_{\mathrm{TM}}$, i.e., a computable function f exists (as defined by F below) such that
$$\langle M, w\rangle \in A_{\mathrm{TM}} \Longleftrightarrow f(\langle M, w\rangle) \in HALT_{\mathrm{TM}}.$$
- $F =$ "On input $\langle M, w\rangle$:
  1. Construct the following machine M'.
     $M' =$ "On input x:
     (a) Run M on x.
     (b) If M accepts, *accept*.
     (c) If M rejects, enter a loop.
  2. Output $\langle M', w\rangle$."

Reducibility and Recognizability

Theorem 18 (5.28)**.** *If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.*

Corollary 19 (5.29)**.** *If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.*

Corollary 20. *If $A \leq_m B$ (i.e., $\overline{A} \leq_m \overline{B}$) and A is not co-Turing-recognizable, then B is not co-Turing-recognizable.*

Note: "A is not co-Turing-recognizable" is the same as "$\overline{A}$ is not Turing-recognizable".

Reducibility and Recognizability (cont.)

Theorem 21 (5.30 Part One)**.** *EQ_{TM} is not Turing-recognizable.*

- We show that A_{TM} reduces to $\overline{EQ_{\text{TM}}}$, i.e., $\overline{A_{\text{TM}}}$ reduces to EQ_{TM}.
- Since $\overline{A_{\text{TM}}}$ is not Turing-recognizable, EQ_{TM} is not Turing-recognizable.
- $F =$ "On input $\langle M, w \rangle$:
  1. Construct the following two machines M_1 and M_2.
     $M1 =$"On any input: reject."
     $M2 =$"On any input: Run M on w. If it accepts, *accept*."
  2. Output $\langle M_1, M_2 \rangle$."

Reducibility and Recognizability (cont.)

Theorem 22 (5.30 Part Two)**.** *EQ_{TM} is not co-Turing-recognizable.*

- We show that A_{TM} reduces to EQ_{TM}.
- Since A_{TM} is not co-Turing-recognizable, EQ_{TM} is not co-Turing-recognizable.
- $G =$ "On input $\langle M, w \rangle$:
  1. Construct the following two machines M_1 and M_2.
     $M1 =$"On any input: *accept*."
     $M2 =$"On any input: Run M on w. If it accepts, *accept*."
  2. Output $\langle M_1, M_2 \rangle$."