Minimization of DFAs

(Based on [Sipser 2013] and [Wikipedia])

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Distinguishable and Indistinguishable Strings

- Let L be a language over Σ (i.e., $L \subseteq \Sigma^*$).
- Two strings x and y in Σ^* are **distinguishable by L** if
  for some string z in Σ^*, exactly one of xz and yz is in L.
- When no such z exists, i.e.,
  for every z in Σ^*, either both of xz and yz or neither of them are in L,
  we say that x and y are **indistinguishable by L**.
- Indistinguishable strings can be regarded as essentially equivalent.

Note: these concepts apply to all languages, not just the regular ones.

Myhill-Nerode Theorem

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- Given a language $L \subseteq \Sigma^*$, define a binary relation R_L over Σ^* as follows:

$$xR_Ly \text{ iff } \forall z \in \Sigma^*(xz \in L \leftrightarrow yz \in L)$$

- R_L can be shown to be an equivalence relation.

Theorem (Myhill-Nerode)

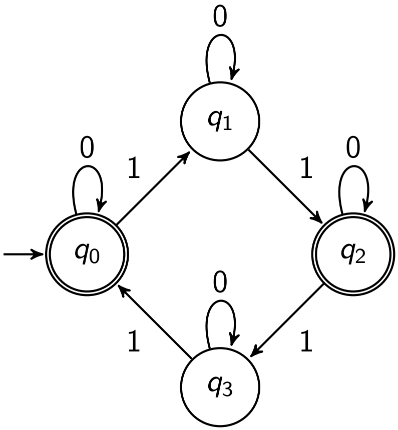
With R_L defined as above, the following are equivalent:

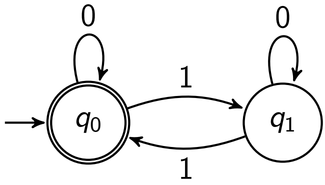
1. *L is regular.*
2. *R_L is of finite index.*

Moreover, the index of R_L equals the number of states in the smallest DFA that recognizes L.

Note: the *index* of an equivalence relation is the number of equivalence classes it induces.

Myhill-Nerode Theorem (cont.)





Both automata recognize the same language (the set of binary strings with an even number of 1's), but the one on the right is clearly smaller and in fact optimal (with the least possible number of states).

Minimization of DFAs

- A DFA $(Q, \Sigma, \delta, q_0, F)$ for L defines an equivalence relation on Σ^* that is a *refinement* of R_L.
- Let $L_q = \{x \in \Sigma^* \mid \delta(q_0, x) = q\}$. Then,
  - for distinct $q, q' \in Q$, $L_q \cap L_{q'} = \emptyset$, and
  - for every $q \in Q$, L_q is contained in an equivalence class of R_L.
- Given a DFA that is not minimum for its language, there must be two distinct states q and q' such that both L_q and $L_{q'}$ are contained in the same equivalence class of R_L and hence may be merged (without affecting the language recognized).

Minimization of DFAs (cont.)

- On the opposite, there are states that must remain separate.
- Apparently, for $q \in F$ and $q' \in Q \setminus F$, L_q and $L_{q'}$ are in different equivalence classes of R_L and hence q and q' must remain separate.
- For any two states, if they can make a transition on the same symbol to two different states that should remain separate, then they should also remain separate; this should be checked repeatedly.
- To minimize a DFA, we may start with the partition $\{F, Q \setminus F\}$ and refine the partition by iteratively checking whether two states in the same block should be separated.

Hopcroft's Minimization Algorithm

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Algorithm Minimization(Q, Σ, δ, F);
begin
    $P := \{F, Q \setminus F\}$; $W := \{F\}$;
    while W not empty **do**
        remove a set A from W;
        for each $c \in \Sigma$ **do**
            $X := \{q \mid \delta(q, c) \in A\}$;
            for each $Y \in P$ s.t. both $X \cap Y$ and $Y \setminus X$ not empty **do**
                replace Y in P by $X \cap Y$ and $Y \setminus X$;
                if $Y \in W$ **then**
                    replace Y in W by $X \cap Y$ and $Y \setminus X$
                else if $|X \cap Y| \leq |Y \setminus X|$ **then**
                        add $X \cap Y$ to W
                    else add $Y \setminus X$ to W
end