

Theory of Computing 2024: More NP-Complete Problems

(Based on [Sipser 2006, 2013])

Yih-Kuen Tsay

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1 More NP-Complete Problems

The Vertex Cover Problem

- A *vertex cover* of an undirected graph G is a subset of the nodes where every edge of G touches one of those nodes.
- $VERTEX_COVER = \{\langle G, k \rangle \mid G \text{ is an undirected graph that has a } k\text{-node vertex cover}\}$.

Theorem 1. $VERTEX_COVER$ is NP-complete.

- We show that $3SAT \leq_P VERTEX_COVER$.

The Vertex Cover Problem (cont.)

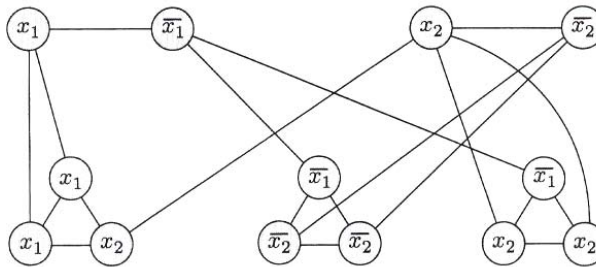


FIGURE 7.45

The graph that the reduction produces from
 $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$

Source: [Sipser 2006]

Note: Let k be $m + 2l$, where m is the number of variables and l the number of clauses in ϕ .

The Hamiltonian Path Problem

Theorem 2. $HAMPATH$ is NP-complete.

We show that $3SAT \leq_P HAMPATH$.

The Hamiltonian Path Problem (cont.)

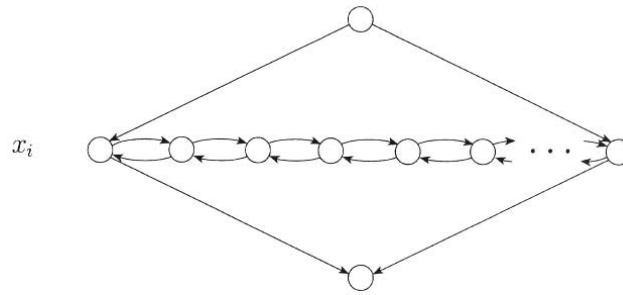


FIGURE 7.47
Representing the variable x_i as a diamond structure

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)



FIGURE 7.48
Representing the clause c_j as a node

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

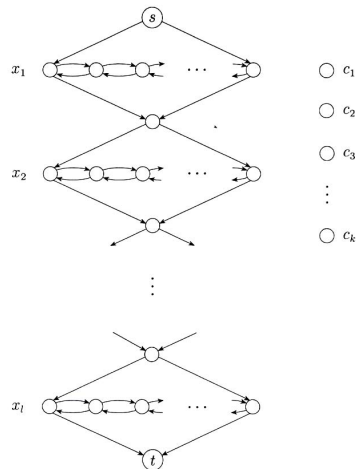


FIGURE 7.49
The high-level structure of G

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

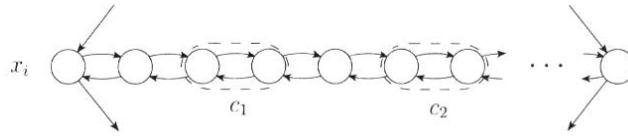


FIGURE 7.50
The horizontal nodes in a diamond structure

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

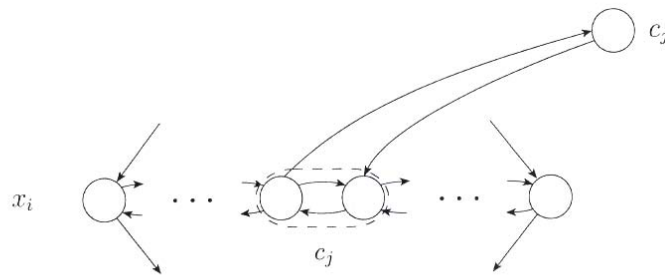


FIGURE 7.51
The additional edges when clause c_j contains x_i

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

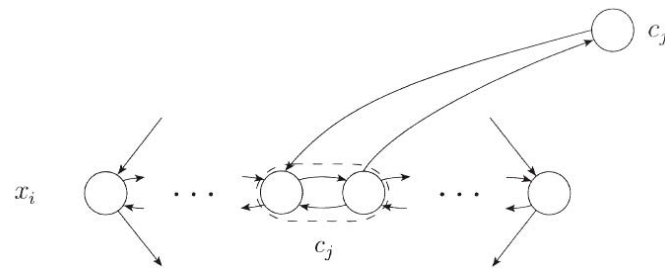


FIGURE 7.52
The additional edges when clause c_j contains \bar{x}_i

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

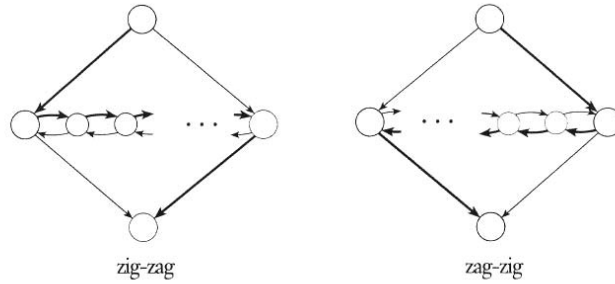


FIGURE 7.53
Zig-zagging and zag-zigging through a diamond, as determined by the satisfying assignment

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

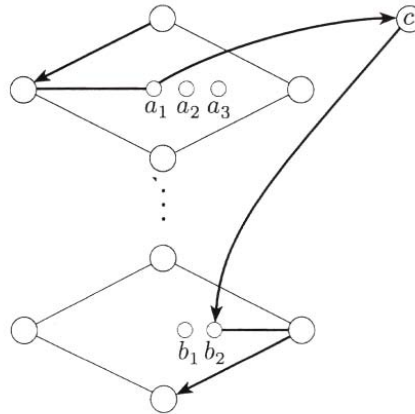


FIGURE 7.54
This situation cannot occur

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

- Let $UHAMPATH$ be the undirected version of the Hamiltonian path problem $HAMPATH$.

Theorem 3. $UHAMPATH$ is NP-complete.

- An input $\langle G, s, t \rangle$ for $HAMPATH$ is mapped to $\langle G', s', t' \rangle$ for $UHAMPATH$ as follows.
- Each node u of G , except for s and t , is replaced by a triple of nodes u^{in} , u^{mid} , and u^{out} in G' .
- Nodes s and t are replaced by node $s^{\text{out}} = s'$ and $t^{\text{in}} = t'$.
- Edges connect u^{mid} with u^{in} and u^{out} .
- An edge connects u^{out} and v^{in} if (u, v) is an edge of G .

The Subset Sum Problem

- $SUBSET_SUM = \{\langle S, t \rangle \mid S = \{x_1, \dots, x_k\} \text{ and for some } \{y_1, \dots, y_l\} \subseteq S, \text{ we have } \sum y_i = t\}$.

Theorem 4. $SUBSET_SUM$ is NP-complete.

- We show that $3SAT \leq_P SUBSET_SUM$.

The Subset Sum Problem (cont.)

	1	2	3	4	...	l	c_1	c_2	...	c_k
y_1	1	0	0	0	...	0	1	0	...	0
z_1	1	0	0	0	...	0	0	0	...	0
y_2		1	0	0	...	0	0	1	...	0
z_2		1	0	0	...	0	1	0	...	0
y_3			1	0	...	0	1	1	...	0
z_3			1	0	...	0	0	0	...	1
\vdots						\ddots	\vdots			\vdots
y_l						1	0	0	...	0
z_l						1	0	0	...	0
g_1							1	0	...	0
h_1							1	0	...	0
g_2								1	...	0
h_2								1	...	0
\vdots										\ddots
g_k										1
h_k										1
t	1	1	1	1	...	1	3	3	...	3

FIGURE 7.57
Reducing 3SAT to SUBSET-SUM

Source: [Sipser 2006]