Searching and Sorting

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Searching a Sorted Sequence

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Problem

Let $x_1, x_2, \cdots, x_n$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Given a real number z, we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.

Binary Search

```
function Find (z, Left, Right) : integer;
begin
    if Left = Right then
        if X[Left] = z then Find := Left
        else Find := 0
    else
        Middle := ⌈(Left+Right)/2⌉;
        if z < X[Middle] then
            Find := Find(z, Left, Middle − 1)
        else
            Find := Find(z, Middle, Right)
end
```

Binary Search (cont.)

Algorithm Binary_Search (X, n, z);
begin
    $Position := Find(z, 1, n)$;
end

Searching a Cyclically Sorted Sequence

Problem

Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).

Cyclic Binary Search

function Cyclic_Find $(Left, Right)$: *integer*;
begin
  if $Left = Right$ **then** $Cyclic\_Find := Left$
  else
    $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
    if $X[Middle] < X[Right]$ **then**
      $Cyclic\_Find := Cyclic\_Find(Left, Middle)$
    else
      $Cyclic\_Find := Cyclic\_Find(Middle + 1, Right)$
end

Cyclic Binary Search (cont.)

Algorithm Cyclic_Binary_Search (X, n);
begin
    $Position := Cyclic\_Find(1, n)$;
end

"Fixpoints"

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Problem

Given a sorted sequence of distinct integers $a_1, a_2, \cdots, a_n$, *determine whether there exists an index* i *such that* $a_i = i$.

A Special Binary Search

function Special_Find $(Left, Right) : integer$;
begin
  if $Left = Right$ **then**
    if $A[Left] = Left$ **then** $Special\_Find := Left$
    else $Special\_Find := 0$
  else
    $Middle := \lceil \frac{Left+Right}{2} \rceil$;
    if $A[Middle] < Middle$ **then**
      $Special\_Find := Special\_Find(Middle + 1, Right)$
    else
      $Special\_Find := Special\_Find(Left, Middle)$
end

A Special Binary Search (cont.)

Algorithm Special_Binary_Search (A, n);
begin
    $Position := Special\_Find(1, n)$;
end

Stuttering Subsequence

Problem

Given two sequences A and B, find the maximal value of i such that B^i is a subsequence of A.

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzxxx$.

Interpolation Search

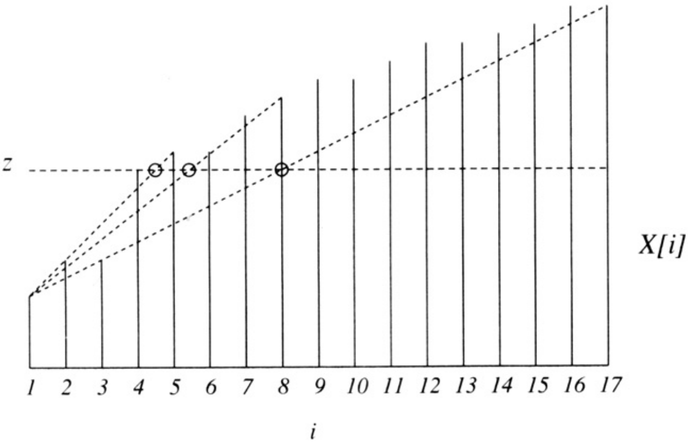

Figure 6.4 Interpolation search.

Source: Manber 1989

Interpolation Search (cont.)

function Int_Find $(z, Left, Right)$: *integer*;
begin
    if $X[Left] = z$ **then** $Int\_Find := Left$
    else if $Left = Right$ or $X[Left] = X[Right]$ **then**
        $Int\_Find := 0$
    else
        $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$;
        if $z < X[Next\_Guess]$ **then**
            $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
        else
            $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end

Interpolation Search (cont.)

Algorithm Interpolation_Search (X, n, z);
begin
    if $z < X[1]$ or $z > X[n]$ **then** $Position := 0$
    else $Position := Int\_Find(z, 1, n)$;
end

Sorting

Problem

Given n numbers $x_1, x_2, \cdots, x_n$, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \leq i_1, i_2, \cdots, i_n \leq n$, such that $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$.

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
  1. Create an empty tree.
  2. Insert the numbers one by one to the tree.
  3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

Radix Sort

```
Algorithm Straight_Radix (X, n, k);
    put all elements of X in a queue GQ;
    for i := 1 to d do
        initialize queue Q[i] to be empty
    for i := k downto 1 do
        while GQ is not empty do
                pop x from GQ;
                d := the i-th digit of x;
                insert x into Q[d];
        for t := 1 to d do
            insert Q[t] into GQ;
    for i := 1 to n do
        pop X[i] from GQ
```

Merge Sort

Algorithm Mergesort (X, n);
begin $M\_Sort(1, n)$ **end**

procedure M_Sort $(Left, Right)$;
begin
  if $Right - Left = 1$ **then**
    if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$
  else if $Left \neq Right$ **then**
      $Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;
      $M\_Sort(Left, Middle - 1)$;
      $M\_Sort(Middle, Right)$;

Merge Sort (cont.)

```
i := Left;  j := Middle;  k := 0;
while (i ≤ Middle − 1) and (j ≤ Right) do
        k := k + 1;
        if X[i] ≤ X[j] then
            TEMP[k] := X[i];  i := i + 1
        else TEMP[k] := X[j];  j := j + 1;
if j > Right then
    for t := 0 to Middle − 1 − i do
        X[Right − t] := X[Middle − 1 − t]
for t := 0 to k − 1 do
    X[Left + t] := TEMP[t]
end
```

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	5	8	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	1	12	15	7	3	13	4	11	16	14
2	5	6	8	1	9	10	12	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	3	13	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	14	16
1	2	5	6	8	9	10	12	3	7	13	15	4	11	14	16
1	2	5	6	8	9	10	12	3	4	7	11	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Figure 6.8 An example of mergesort. The first row is in the initial order. Each row illustrates either an exchange operation or a merge. The numbers that are involved in the current operation are circled.

Source: Manber 1989

Quick Sort

Algorithm Quicksort (X, n);
begin
    $Q\_Sort(1, n)$
end

procedure Q_Sort $(Left, Right)$;
begin
    if $Left < Right$ **then**
        $Partition(X, Left, Right)$;
        $Q\_Sort(Left, Middle - 1)$;
        $Q\_Sort(Middle + 1, Right)$
end

Quick Sort (cont.)

Algorithm Partition $(X, Left, Right)$;
begin
$pivot := X[left]$;
$L := Left$; $R := Right$;
while $L < R$ **do**
while $X[L] \leq pivot$ and $L \leq Right$ **do** $L := L + 1$;
while $X[R] > pivot$ and $R \geq Left$ **do** $R := R - 1$;
if $L < R$ **then** $swap(X[L], X[R])$;
$Middle := R$;
$swap(X[Left], X[Middle])$
end

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	4	5	10	9	12	1	15	7	3	13	8	11	16	14
6	2	4	5	3	9	12	1	15	7	10	13	8	11	16	14
6	2	4	5	3	1	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14

Figure 6.10 Partition of an array around the pivot 6.

Source: Manber 1989

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	8	9	11	7	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	11	9	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	10	9	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	14	⑮	16

Figure 6.12 An example of quicksort. The first line is the initial input. A new pivot is selected in each line. The pivots are circled. When a single number appears between two pivots it is obviously in the right position.

Source: Manber 1989

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

  The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \tfrac{1}{n}\sum_{i=1}^{n}(T(i-1) + T(n-i)) \\ &= n - 1 + \tfrac{1}{n}\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n}\sum_{i=1}^{n} T(n-i) \\ &= n - 1 + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \tfrac{2}{n}\sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

Heap Sort

Algorithm Heapsort (A, n);
begin
    Build_Heap(A);
    for $i := n$ **downto** 2 **do**
        swap$(A[1], A[i])$;
        Rearrange_Heap$(i - 1)$
end

Heap Sort

```
procedure Rearrange_Heap (k);
begin
    parent := 1;
    child := 2;
    while child ≤ k − 1 do
        if A[child] < A[child + 1] then
            child := child + 1;
        if A[child] > A[parent] then
            swap(A[parent], A[child]);
            parent := child;
            child := 2 ∗ child
        else child := k
end
```

Heap Sort (cont.)

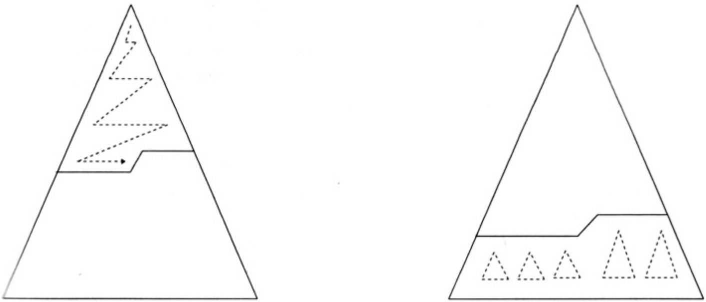

Figure 6.14 Top down and bottom up heap construction.

Source: Manber 1989

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	(14)	15	7	3	13	4	11	16	(1)
2	6	8	5	10	9	(16)	14	15	7	3	13	4	11	(12)	1
2	6	8	5	10	(13)	16	14	15	7	3	(9)	4	11	12	1
2	6	8	5	10	13	16	14	15	7	3	9	4	11	12	1
2	6	8	(15)	10	13	16	14	(5)	7	3	9	4	11	12	1
2	6	(16)	15	10	13	(12)	14	5	7	3	9	4	11	(8)	1
2	(15)	16	(14)	10	13	12	(6)	5	7	3	9	4	11	8	1
(16)	15	(13)	14	10	(9)	12	6	5	7	3	(2)	4	11	8	1

Figure 6.15 An example of building a heap bottom up. The numbers on top are the indices. The circled numbers are those that have been exchanged on that step.

Source: Manber 1989

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

Theorem (Theorem 6.1)

Every decision-tree algorithm for sorting has height $\Omega(n \log n)$.

Order Statistics: Minimum and Maximum

Problem

Find the maximum and minimum elements in a given sequence.

Order Statistics: Kth-Smallest

Problem

Given a sequence $S = x_1, x_2, \cdots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the kth-smallest element in S.

Order Statistics: Kth-Smallest (cont.)

procedure Select $(Left, Right, k)$;
begin
  if $Left = Right$ **then**
    $Select := Left$
  else $Partition(X, Left, Right)$;
    let Middle be the output of Partition;
    if $Middle - Left + 1 \geq k$ **then**
      $Select(Left, Middle, k)$
    else
      $Select(Middle + 1, Right, k - (Middle - Left + 1))$
end

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

procedure Select $(Left, Right, k)$;
begin
  if $Left = Right$ **then**
    $Select := Left$
  else $Partition(X, Left, Right)$;
    let Middle be the output of Partition;
    if $Middle \geq k$ **then**
      $Select(Left, Middle, k)$
    else
      $Select(Middle + 1, Right, k)$
end

Order Statistics: Kth-Smallest (cont.)

Algorithm Selection (X, n, k);
begin
    if $(k < 1)$ or $(k > n)$ **then** *print* *"error"*
    else $S := Select(1, n, k)$
end

Finding a Majority

Problem

Given a sequence of numbers, find the majority in the sequence or determine that none exists.

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Finding a Majority (cont.)

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Algorithm Majority (X, n);
begin
    $C := X[1];\ \ M := 1;$
    for $i := 2$ **to** n **do**
        if $M = 0$ **then**
            $C := X[i];\ \ M := 1$
        else
            if $C = X[i]$ **then** $M := M + 1$
            else $M := M - 1;$

Finding a Majority (cont.)

if $M = 0$ **then** $Majority := -1$
else
    $Count := 0;$
    for $i := 1$ **to** n **do**
        if $X[i] = C$ **then** $Count := Count + 1;$
    if $Count > n/2$ **then** $Majority := C$
    else $Majority := -1$
end