

# Mathematical Induction

(Based on [Manber 1989])

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# The Standard Induction Principle

- Let  $T$  be a theorem that includes a parameter  $n$  whose value can be any natural number.
- Here, natural numbers are positive integers, i.e.,  $1, 2, 3, \dots$ , excluding  $0$  (sometimes we may include  $0$ ).
- To prove  $T$ , it suffices to prove the following two conditions:
  - $T$  holds for  $n = 1$ . (Base case)
  - For every  $n > 1$ , if  $T$  holds for  $n - 1$ , then  $T$  holds for  $n$ . (Inductive step)
- The assumption in the inductive step that  $T$  holds for  $n - 1$  is called the *induction hypothesis*.

# A Starter

## Theorem (2.1)

For all natural numbers  $x$  and  $n$ ,  $x^n - 1$  is divisible by  $x - 1$ .

## Proof.

(Suggestion: try to follow the structure of this proof when you present a proof by induction.)

The proof is **by induction on  $n$** .

**Base case:**  $x - 1$  is trivially divisible by  $x - 1$ .

**Inductive step:**  $x^n - 1 = x(x^{n-1} - 1) + (x - 1)$ .  $x^{n-1} - 1$  is divisible by  $x - 1$  **from the induction hypothesis** and  $x - 1$  is divisible by  $x - 1$ . Hence,  $x^n - 1$  is divisible by  $x - 1$ . □

Note:  $a$  is divisible by  $b$  if there exists an integer  $c$  such that  $a = b \times c$ .

# Variants of Induction Principle

## Theorem

*If a statement  $P$ , with a parameter  $n$ , is true for  $n = 1$ , and if, for every  $n \geq 1$ , the truth of  $P$  for  $n$  implies its truth for  $n + 1$ , then  $P$  is true for all natural numbers.*

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## Theorem (Strong Induction)

*If a statement  $P$ , with a parameter  $n$ , is true for  $n = 1$ , and if, for every  $n > 1$ , the truth of  $P$  for all natural numbers  $< n$  implies its truth for  $n$ , then  $P$  is true for all natural numbers.*

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## Theorem

*If a statement  $P$ , with a parameter  $n$ , is true for  $n = 1$  and for  $n = 2$ , and if, for every  $n > 2$ , the truth of  $P$  for  $n - 2$  implies its truth for  $n$ , then  $P$  is true for all natural numbers.*

# Design by Induction: First Glimpse

## Problem

Given two *sorted* arrays  $A[1..m]$  and  $B[1..n]$  of positive integers, find their *smallest common element*; returns 0 if no common element is found.

- 🌐 Assume the elements of each array are in **ascending** order.
- 🌐 **Obvious solution:** take one element at a time from  $A$  and find out if it is also in  $B$  (or the other way around).

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- 🌐 Assume the elements of each array are in **ascending** order.
- 🌐 **Obvious solution**: take one element at a time from  $A$  and find out if it is also in  $B$  (or the other way around).
- 🌐 How efficient is this solution?
- 🌐 Can we do better?



# Design by Induction: First Glimpse (cont.)

- There are  $m + n$  elements to begin with.
- Can we pick out one element such that either (1) it is the element we look for or (2) it can be ruled out from subsequent searches?
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- In the second case, we are left with the same problem but with  $m + n - 1$  elements?
- Idea:** compare the current first elements of  $A$  and  $B$ .
  - If they are equal, then we are done.
  - If not, the smaller one cannot be the smallest common element.

# Design by Induction: First Glimpse (cont.)



Below is the complete solution:

## Algorithm

```
Algorithm  $SCE(A, m, B, n) : integer;$   
begin  
  if  $m = 0$  or  $n = 0$  then  $SCE := 0;$   
  if  $A[1] = B[1]$  then  
     $SCE := A[1];$   
  else if  $A[1] < B[1]$  then  
     $SCE := SCE(A[2..m], m - 1, B, n);$   
  else  $SCE := SCE(A, m, B[2..n], n - 1);$   
end
```

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  - ☀️ various manipulations of the objects become functions on the corresponding mathematical structures.
- 🌐 Many mathematical structures are naturally defined by induction.
- 🌐 Functions on inductive structures are also naturally defined by induction (recursion).


# Another Simple Example

## Theorem (2.4)

If  $n$  is a natural number and  $1 + x > 0$ , then  $(1 + x)^n \geq 1 + nx$ .

 Below are the key steps:

$$\begin{aligned}(1 + x)^{n+1} &= (1 + x)(1 + x)^n \\ &\quad \{\text{induction hypothesis and } 1 + x > 0\} \\ &\geq (1 + x)(1 + nx) \\ &= 1 + (n + 1)x + nx^2 \\ &\geq 1 + (n + 1)x\end{aligned}$$

 The main point here is that we should be clear about how conditions listed in the theorem are used.



# Proving vs. Computing

## Theorem (2.2)

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

- 🌐 This can be easily proven by induction.
- 🌐 Key steps:  $1 + 2 + \cdots + n + (n + 1) = \frac{n(n+1)}{2} + (n + 1) = \frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}.$

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- 🌐 Induction seems to be useful only if we already know the sum.
- 🌐 What if we are asked to **compute** the sum of a series?
- 🌐 Let's try  $8 + 13 + 18 + 23 + \cdots + (3 + 5n).$

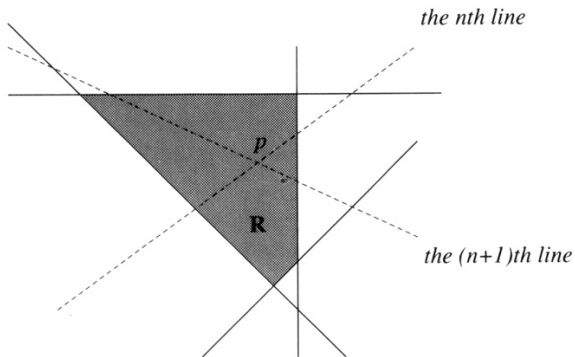
# Proving vs. Computing (cont.)

- 🌐 **Idea:** guess and then verify by an inductive proof!
- 🌐 The sum should be of the form  $an^2 + bn + c$ .
- 🌐 By checking  $n = 1, 2,$  and  $3,$  we get  $\frac{5}{2}n^2 + \frac{11}{2}n$ .
- 🌐 Verify this for all  $n,$  i.e., the following theorem, by induction.

## Theorem (2.3)

$$8 + 13 + 18 + 23 + \cdots + (3 + 5n) = \frac{5}{2}n^2 + \frac{11}{2}n.$$

# Counting Regions



**Figure 2.1**  $n + 1$  lines in general position.

Source: [Manber 1989].

# Counting Regions (cont.)

## Theorem (2.5)

*The number of regions in the plane formed by  $n$  lines in general position is  $\frac{n(n+1)}{2} + 1$ .*

A set of lines are in **general position** if (1) no two lines are parallel and (2) no three lines intersect at a common point.

# Counting Regions (cont.)

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A set of lines are in **general position** if (1) no two lines are parallel and (2) no three lines intersect at a common point.

🌐 We observe that  $\frac{n(n+1)}{2} = 1 + 2 + \cdots + n$ .

🌐 So, it suffices to prove the following:

## Lemma

*Adding one more line (the  $n$ -th line) to  $n - 1$  lines in general position in the plane increases the number of regions by  $n$ .*

# A Summation Problem

$$\begin{aligned}1 &= 1 \\3 + 5 &= 8 \\7 + 9 + 11 &= 27 \\13 + 15 + 17 + 19 &= 64 \\21 + 23 + 25 + 27 + 29 &= 125\end{aligned}$$

## Theorem

*The sum of row  $n$  in the triangle is  $n^3$ .*



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## Theorem

*The sum of row  $n$  in the triangle is  $n^3$ .*

Examine the difference between rows  $i + 1$  and  $i \dots$


## Lemma

*The last number in row  $n + 1$  is  $n^2 + 3n + 1$ .*

# A Simple Inequality

## Theorem (2.7)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 1, \text{ for all } n \geq 1.$$

 There are at least two ways to select  $n$  terms from  $n + 1$  terms.

1.  $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}) + \frac{1}{2^{n+1}}$ .

# A Simple Inequality

## Theorem (2.7)

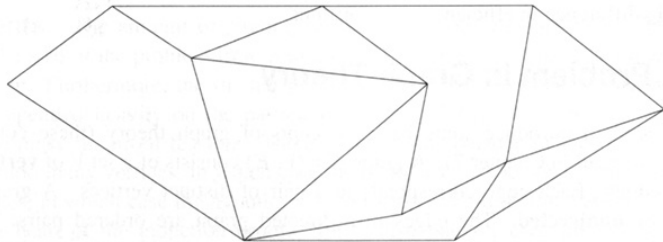
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🌐 There are at least two ways to select  $n$  terms from  $n + 1$  terms.

1.  $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}) + \frac{1}{2^{n+1}}$ .
2.  $\frac{1}{2} + (\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}})$ .

🌐 The second one leads to a successful inductive proof:

$$\begin{aligned} & \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \right) \\ &= \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} \right) \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$



**Figure 2.2** A planar map with 11 vertices, 19 edges, and 10 faces.

Source: [Manber 1989].

# Euler's Formula (cont.)

## Theorem (2.8)

*The number of vertices ( $V$ ), edges ( $E$ ), and faces ( $F$ ) in an arbitrary connected planar graph are related by the formula  $V + F = E + 2$ .*

# Euler's Formula (cont.)

## Theorem (2.8)

*The number of vertices ( $V$ ), edges ( $E$ ), and faces ( $F$ ) in an arbitrary connected planar graph are related by the formula  $V + F = E + 2$ .*

The proof is by induction on the number of faces.  
Base case: graphs with only one face are **trees** ...

## Lemma

*A tree with  $n$  vertices has  $n - 1$  edges.*

Inductive step: for a graph with more than one faces, there must be a **cycle** in the graph. Remove one edge from the cycle ...

# Gray Codes

🌐 A **Gray code** (after Frank Gray) for  $n$  objects is a binary-encoding scheme for naming the  $n$  objects such that the  $n$  names can be arranged in a *circular* list where *any two adjacent names differ by only one bit*.

🌐 Examples:

☀ 00, 01, 11, 10

☀ 000, 001, 011, 010, 110, 111, 101, 100

☀ 000, 001, 011, 111, 101, 100

# Gray Codes (cont.)

## Theorem (2.10)

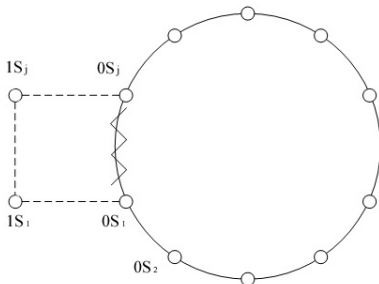
*There exist Gray codes of length  $\frac{k}{2}$  for any positive even integer  $k$ .*



# Gray Codes (cont.)

## Theorem (2.10)

*There exist Gray codes of length  $\frac{k}{2}$  for any positive even integer  $k$ .*



**Figure 2.3** Constructing a Gray code of size  $2k$

Source: [Manber 1989] (adapted).

Note:  $j$  in the figure equals  $2(k - 1)$  and hence  $j + 2$  equals  $2k$ .

# Gray Codes (cont.)

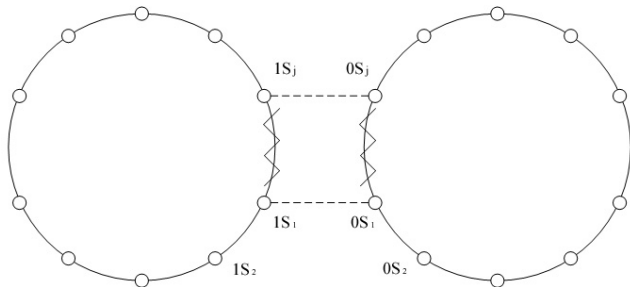
## Theorem (2.10+)

*There exist Gray codes of length  $\log_2 k$  for any positive integer  $k$  that is a power of 2.*

# Gray Codes (cont.)

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**Figure 2.4** Constructing a Gray code from two smaller ones

Source: [Manber 1989] (adapted).

## Gray Codes (cont.)

- 🌐 00, 01, 11, 10 (for  $2^2$  objects)
- 🌐 000, 001, 011, 010 (add a 0)
- 🌐 100, 101, 111, 110 (add a 1)
- 🌐 Combine the preceding two codes (read the second in reversed order):  
000, 001, 011, 010, 110, 111, 101, 100 (for  $2^3$  objects)

# Gray Codes (cont.)

## Theorem (2.11–)

*There exist Gray codes of length  $\lceil \log_2 k \rceil$  for any positive even integer  $k$ .*

## Gray Codes (cont.)

### Theorem (2.11–)

*There exist Gray codes of length  $\lceil \log_2 k \rceil$  for any positive even integer  $k$ .*

To generalize the result and ease the proof, we allow a Gray code to be *open* where the last name and the first name may differ by more than one bit.

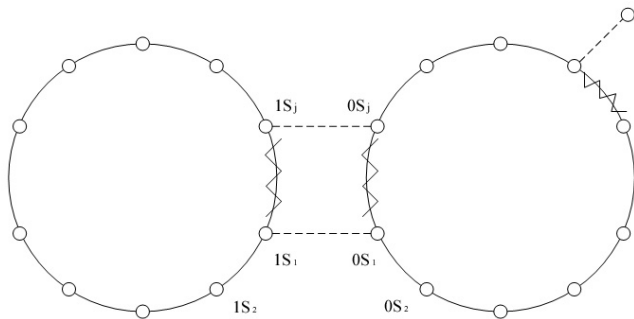
### Theorem (2.11)

*There exist Gray codes of length  $\lceil \log_2 k \rceil$  for any positive integer  $k \geq 2$ . The Gray codes for the *even* values of  $k$  are *closed*, and the Gray codes for *odd* values of  $k$  are *open*.*

# Gray Codes (cont.)

- 00, 01, 11 (open Gray code for 3 objects)
- 000, 001, 011 (add a 0)
- 100, 101, 111 (add a 1)
- Combine the preceding two codes (read the second in reversed order):  
000, 001, 011, 111, 101, 100 (closed Gray code for 6 objects)

# Gray Codes (cont.)



**Figure 2.5** Constructing an open Gray code

Source: [Manber 1989] (adapted).



# Arithmetic vs. Geometric Mean

## Theorem (2.13)

If  $x_1, x_2, \dots, x_n$  are all positive numbers, then

$$(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

# Arithmetic vs. Geometric Mean

## Theorem (2.13)

If  $x_1, x_2, \dots, x_n$  are all positive numbers, then

$$(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

First use the standard induction to prove the case of powers of 2 and then use the reversed induction principle below to prove for all natural numbers.

## Theorem (Reversed Induction Principle)

If a statement  $P$ , with a parameter  $n$ , is true for an *infinite subset* of the natural numbers, and if, for every  $n > 1$ , the truth of  $P$  for  $n$  implies its truth for  $n - 1$ , then  $P$  is true for all natural numbers.

# Arithmetic vs. Geometric Mean (cont.)

- 🌐 For all powers of 2, i.e.,  $n = 2^k$ ,  $k \geq 1$ : by induction on  $k$ .
- 🌐 Base case:  $(x_1 x_2)^{\frac{1}{2}} \leq \frac{x_1 + x_2}{2}$ , squaring both sides . . . .

# Arithmetic vs. Geometric Mean (cont.)

- For all powers of 2, i.e.,  $n = 2^k$ ,  $k \geq 1$ : by induction on  $k$ .
- Base case:  $(x_1 x_2)^{\frac{1}{2}} \leq \frac{x_1 + x_2}{2}$ , squaring both sides . . . .
- Inductive step:

$$(x_1 x_2 \cdots x_{2^{k+1}})^{\frac{1}{2^{k+1}}}$$

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- Base case:  $(x_1 x_2)^{\frac{1}{2}} \leq \frac{x_1 + x_2}{2}$ , squaring both sides . . . .
- Inductive step:

$$\begin{aligned} & (x_1 x_2 \cdots x_{2^{k+1}})^{\frac{1}{2^{k+1}}} \\ = & \left[ (x_1 x_2 \cdots x_{2^k})^{\frac{1}{2^k}} \right]^{\frac{1}{2}} \end{aligned}$$

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- Base case:  $(x_1 x_2)^{\frac{1}{2}} \leq \frac{x_1 + x_2}{2}$ , squaring both sides . . . .
- Inductive step:

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 & (x_1 x_2 \cdots x_{2^{k+1}})^{\frac{1}{2^{k+1}}} \\
 = & \left[ (x_1 x_2 \cdots x_{2^k})^{\frac{1}{2^k}} \right]^{\frac{1}{2}} \\
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 \leq & \frac{(x_1 x_2 \cdots x_{2^k})^{\frac{1}{2^k}} + (x_{2^k+1} x_{2^k+2} \cdots x_{2^{k+1}})^{\frac{1}{2^k}}}{2}, \text{ from the base case}
 \end{aligned}$$

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 \leq & \frac{\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + x_{2^k+2} + \cdots + x_{2^{k+1}}}{2^k}}{2}, \text{ from the Ind. Hypo.}
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 = & \frac{x_1 + x_2 + \cdots + x_{2^{k+1}}}{2^{k+1}}
 \end{aligned}$$

# Arithmetic vs. Geometric Mean (cont.)

- 🌐 For all natural numbers: by reversed induction on  $n$ .
- 🌐 Base case: the theorem holds for all powers of 2.

# Arithmetic vs. Geometric Mean (cont.)

- For all natural numbers: by reversed induction on  $n$ .
- Base case: the theorem holds for all powers of 2.
- Inductive step: observe that

$$\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} = \frac{x_1 + x_2 + \cdots + x_{n-1} + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}}{n}.$$

# Arithmetic vs. Geometric Mean (cont.)

$$\left(x_1 x_2 \cdots x_{n-1} \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)\right)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_{n-1} + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}}{n}$$

(from the Ind. Hypo.)

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$$\left(x_1 x_2 \cdots x_{n-1} \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)\right) \leq \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)^n$$

# Arithmetic vs. Geometric Mean (cont.)

$$(x_1 x_2 \cdots x_{n-1} \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right))^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_{n-1} + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}}{n}$$

(from the Ind. Hypo.)

$$(x_1 x_2 \cdots x_{n-1} \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right))^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}$$

$$(x_1 x_2 \cdots x_{n-1} \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)) \leq \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)^n$$

$$(x_1 x_2 \cdots x_{n-1}) \leq \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)^{n-1}$$

# Arithmetic vs. Geometric Mean (cont.)

$$(x_1 x_2 \cdots x_{n-1} \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right))^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_{n-1} + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}}{n}$$

(from the Ind. Hypo.)

$$(x_1 x_2 \cdots x_{n-1} \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right))^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}$$

$$(x_1 x_2 \cdots x_{n-1} \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)) \leq \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)^n$$

$$(x_1 x_2 \cdots x_{n-1}) \leq \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)^{n-1}$$

$$(x_1 x_2 \cdots x_{n-1})^{\frac{1}{n-1}} \leq \left( \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)$$



# Loop Invariants

- 🌐 An *invariant* at some point of a program is an assertion that holds whenever execution of the program reaches that point.
- 🌐 Invariants are a bridge between the **static text** of a program and its **dynamic computation**.

# Loop Invariants

- 🌐 An *invariant* at some point of a program is an assertion that holds whenever execution of the program reaches that point.
- 🌐 Invariants are a bridge between the **static text** of a program and its **dynamic computation**.
- 🌐 An invariant at the front of a while loop is called a *loop invariant* of the while loop.
- 🌐 A loop invariant is formally established by induction.
  - ☀ **Base case**: the assertion holds right before the loop starts.
  - ☀ **Inductive step**: assuming the assertion holds before the  $i$ -th iteration ( $i \geq 1$ ), it holds again after the iteration.

# Number Conversion

## Algorithm

**Algorithm Convert\_to\_Binary( $n$ );**

**begin**

$t := n;$

$k := 0;$

**while**  $t > 0$  **do**

$k := k + 1;$

$b[k] := t \bmod 2;$

$t := t \operatorname{div} 2;$

**end**

# Number Conversion (cont.)

## Theorem (2.14)

*When Algorithm Convert\_to\_Binary terminates, the binary representation of  $n$  is stored in the array  $b$ .*

## Lemma

*If  $m$  is the integer represented by the binary array  $b[1..k]$ , then  $n = t \cdot 2^k + m$  is a loop invariant of the while loop.*

See separate handout for a detailed proof.