Algorithms 2017: Mathematical Induction

(Based on [Manber 1989])

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1 Induction Principles

The Standard Induction Principle

- Let T be a theorem that includes a parameter n whose value can be any natural number.
- Here, natural numbers are positive integers, i.e., 1, 2, 3, ..., excluding 0 (sometimes we may include 0).
- To prove T, it suffices to prove the following two conditions:
  - T holds for $n = 1$. (Base case)
  - For every $n > 1$, if T holds for $n - 1$, then T holds for n. (Inductive step)
- The assumption in the inductive step that T holds for $n - 1$ is called the *induction hypothesis*.

A Simple Proof by Induction

Theorem 1 (2.1)**.** *For all natural numbers x and n, $x^n - 1$ is divisible by $x - 1$.*

Proof. (Suggestion: try to follow the structure of this proof when you present a proof by induction.)

The proof is by induction on n.

Base case ($n = 1$): $x - 1$ is trivially divisible by $x - 1$.

Inductive step ($n > 1$): $x^n - 1 = x(x^{n-1} - 1) + (x - 1)$. $x^{n-1} - 1$ is divisible by $x - 1$ *from the induction hypothesis* and $x - 1$ is divisible by $x - 1$. Hence, $x^n - 1$ is divisible by $x - 1$. □

Note: a is divisible by b if there exists an integer c such that $a = b \times c$.

Variants of Induction Principle

Theorem 2. *If a statement P, with a parameter n, is true for $n = 1$, and if, for every $n \geq 1$, the truth of P for n implies its truth for $n + 1$, then P is true for all natural numbers.*

Theorem 3 (Strong Induction)**.** *If a statement P, with a parameter n, is true for $n = 1$, and if, for every $n > 1$, the truth of P for all natural numbers $< n$ implies its truth for n, then P is true for all natural numbers.*

Theorem 4. *If a statement P, with a parameter n, is true for $n = 1$ and for $n = 2$, and if, for every $n > 2$, the truth of P for $n - 2$ implies its truth for n, then P is true for all natural numbers.*

2 Design by Induction

Design by Induction: First Glimpse

- The selection sort, for instance, can be seen as constructed using design by induction:
  1. When there is only one element, we are done.
  2. When there are n (> 1) elements, we
     (a) select the largest element,
     (b) sort the remaining $n - 1$ elements, and
     (c) append the largest element to the sorted $n - 1$ elements.
- This looks simple enough, but the selection sort isn't very efficient.
- How can we obtain a more efficient algorithm via design by induction?
- To see the power of design by induction, let's look at a less familiar example.

Design by Induction: First Glimpse (cont.)

Problem 5. *Given two sorted arrays $A[1..m]$ and $B[1..n]$ of positive integers, find their smallest common element; returns 0 if no common element is found.*

- Assume the elements of each array are in ascending order.
- **Obvious solution**: take one element at a time from A and find out if it is also in B (or the other way around).
- How efficient is this solution?
- Can we do better?

Design by Induction: First Glimpse (cont.)

- There are $m + n$ elements to begin with.
- Can we pick out one element such that either (1) it is the element we look for or (2) it can be ruled out from subsequent searches?
- In the second case, we are left with the same problem but with $m + n - 1$ elements?
- **Idea**: compare the current first elements of A and B.
  1. If they are equal, then we are done.
  2. If not, the smaller one cannot be the smallest common element.

Design by Induction: First Glimpse (cont.)

Below is the complete solution:

Algorithm SCE(A, m, B, n) : *integer*;
begin
  if $m = 0$ or $n = 0$ **then** $SCE := 0$;
  if $A[1] = B[1]$ **then**
    $SCE := A[1]$;
  else if $A[1] < B[1]$ **then**
      $SCE := SCE(A[2..m], m - 1, B, n)$;
  else $SCE := SCE(A, m, B[2..n], n - 1)$;
end

Why Induction Works

- Computations carried out by a computer/machine can, in essence, be understood as mathematical functions.
- To solve practical problems with computers,
  - objects/things in a practical domain must be modeled as (mostly discrete) mathematical structures/sets, and
  - various manipulations of the objects become functions on the corresponding mathematical structures.
- Many mathematical structures are naturally defined by induction.
- Functions on inductive structures are also naturally defined by induction (recursion).

Recursively/Inductively-Defined Sets

- The natural numbers (including 0):
  1. Base case: 0 is a natural number.
  2. Inductive step: if n is a natural number, then $n + 1$ is also a natural number.
- Binary trees:
  1. Base case: the empty tree is a binary tree.
  2. Inductive step: if L and R are binary trees, then a node with L and R as the left and the right children is also a binary tree.
- Nonempty binary trees:
  1. Base case: a single root node (without any child) is a binary tree.
  2. Inductive step: if L and R are binary trees, then a node with L as the left child and/or R as the right child is also a binary tree.

Structural Induction

- Structural induction is a generalization of mathematical induction on the natural numbers.
- It is used to prove that some proposition $P(x)$ holds for all x of some sort of recursively/inductively defined structure such as binary trees.
- Proof by structural induction:
  1. Base case: the proposition holds for all the minimal structures.
  2. Inductive step: if the proposition holds for the immediate substructures of a certain structure S, then it also holds for S.

3 Proofs by Induction

Another Simple Example

Theorem 6 (2.4)**.** *If n is a natural number and $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$.*

- Below are the key steps:

$$\begin{aligned} (1+x)^{n+1} &= (1+x)(1+x)^n \\ &\quad \{\text{induction hypothesis and } 1+x>0\} \\ &\geq (1+x)(1+nx) \\ &= 1+(n+1)x+nx^2 \\ &\geq 1+(n+1)x \end{aligned}$$

- The main point here is that we should be clear about how conditions listed in the theorem are used.

3.1 Proving vs. Computing

Proving vs. Computing

Theorem 7 (2.2)**.** $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

- This can be easily proven by induction.
- Key steps: $1 + 2 + \cdots + n + (n + 1) = \frac{n(n+1)}{2} + (n + 1) = \frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$.
- Induction seems to be useful only if we already know the sum.
- What if we are asked to compute the sum of a series?
- Let's try $8 + 13 + 18 + 23 + \cdots + (3 + 5n)$.

Proving vs. Computing (cont.)

- **Idea**: guess and then verify by an inductive proof!
- The sum should be of the form $an^2 + bn + c$.
- By checking $n = 1$, 2, and 3, we get $\frac{5}{2}n^2 + \frac{11}{2}n$.
- Verify this for all n, i.e., the following theorem, by induction.

Theorem 8 (2.3)**.** $8 + 13 + 18 + 23 + \cdots + (3 + 5n) = \frac{5}{2}n^2 + \frac{11}{2}n$.

3.2 Counting Regions

Counting Regions

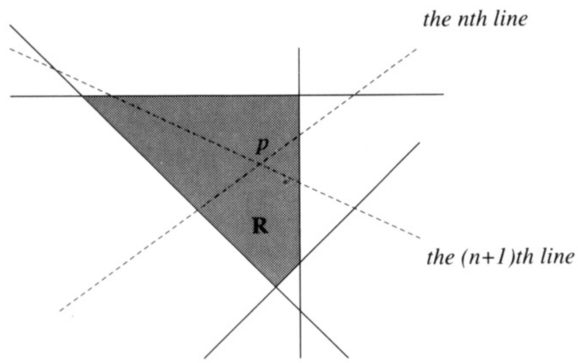


Figure 2.1 $n + 1$ lines in general position.

Source: [Manber 1989].

Counting Regions (cont.)

Theorem 9 (2.5)**.** *The number of regions in the plane formed by n lines in general position is* $\frac{n(n+1)}{2} + 1$.

A set of lines are in **general position** if (1) no two lines are parallel and (2) no three lines intersect at a common point.

- We observe that $\frac{n(n+1)}{2} = 1 + 2 + \cdots + n$.
- So, it suffices to prove the following:

Lemma 10. *Adding one more line (the n-th line) to $n - 1$ lines in general position in the plane increases the number of regions by n.*

3.3 A Summation Problem

A Summation Problem

$$\begin{array}{ccc} 1 & = & 1 \\ 3+5 & = & 8 \\ 7+9+11 & = & 27 \\ 13+15+17+19 & = & 64 \\ 21+23+25+27+29 & = & 125 \end{array}$$

Theorem 11. *The sum of row n in the triangle is* n^3.

Examine the difference between rows $i + 1$ and i ...

Lemma 12. *The last number in row $n + 1$ is* $n^2 + 3n + 1$.

A Simple Inequality

Theorem 13 (2.7)**.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 1$*, for all* $n \geq 1$.

- There are at least two ways to select n terms from $n + 1$ terms.

  1. $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}) + \frac{1}{2^{n+1}}$.

2. $\frac{1}{2} + (\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}})$.

- The second one leads to a successful inductive proof:

$$\begin{aligned} & \frac{1}{2} + (\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}) \\ = & \frac{1}{2} + \frac{1}{2}(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}) \\ < & \frac{1}{2} + \frac{1}{2} \\ = & 1 \end{aligned}$$

3.4 Euler's Formula

Euler's Formula

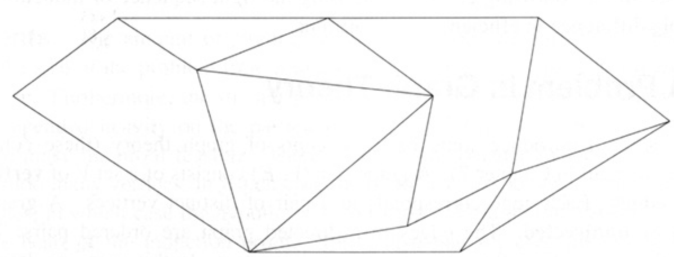


Figure 2.2 A planar map with 11 vertices, 19 edges, and 10 faces.

Source: [Manber 1989].

Euler's Formula (cont.)

Theorem 14 (2.8)**.** *The number of vertices (V), edges (E), and faces (F) in an arbitrary connected planar graph are related by the formula $V + F = E + 2$.*

The proof is by induction on the number of faces.
Base case: graphs with only one face are trees ...

Lemma 15. *A tree with n vertices has $n - 1$ edges.*

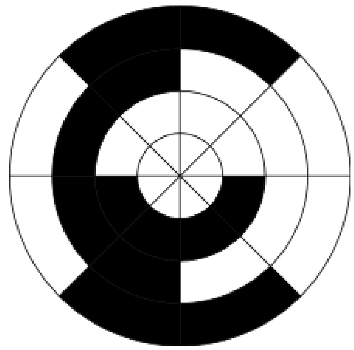
Inductive step: for a graph with more than one faces, there must be a cycle in the graph. Remove one edge from the cyle ...

3.5 Gray Codes

Gray Codes

- A **Gray code** (after Frank Gray) for n objects is a binary-encoding scheme for naming the n objects such that the n names can be arranged in a *circular* list where *any two adjacent names, or code words, differ by only one bit*.
- Examples:
  - 00, 01, 11, 10
  - 000, 001, 011, 010, 110, 111, 101, 100
  - 000, 001, 011, 111, 101, 100

A Gray Code in Picture



A rotary encoder using a 3-bit Gray code.

Source: Wikipedia.

Gray Codes (cont.)

Theorem 16 (2.10)**.** *There exist Gray codes of length* $\frac{k}{2}$ *for any positive even integer* k*.*

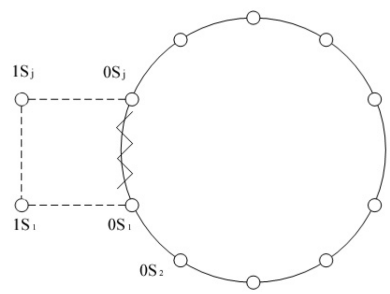


Figure 2.3 Constructing a Gray code of size *2k*

Source: [Manber 1989] (adapted).

Note: j in the figure equals $2(k-1)$ and hence $j+2$ equals $2k$.

Gray Codes (cont.)

Theorem 17 (2.10+)**.** *There exist Gray codes of length* $\log_2 k$ *for any positive integer* k *that is a power of* 2*.*

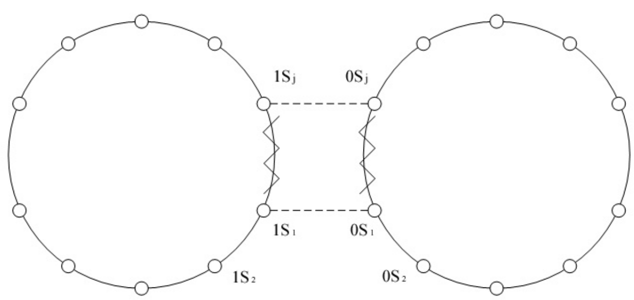


Figure 2.4 Constructing a Gray code from two smaller ones

Source: [Manber 1989] (adapted).

Gray Codes (cont.)

- 00, 01, 11, 10 (for 2^2 objects)
- 000, 001, 011, 010 (add a 0)
- 100, 101, 111, 110 (add a 1)
- Combine the preceding two codes (read the second in reversed order): 000, 001, 011, 010, 110, 111, 101, 100 (for 2^3 objects)

Gray Codes (cont.)

Theorem 18 (2.11−). *There exist Gray codes of length $\lceil \log_2 k \rceil$ for any positive even integer k.*

To generalize the result and ease the proof, we allow a Gray code to be *open* where the last name and the first name may differ by more than one bit.

Theorem 19 (2.11). *There exist Gray codes of length $\lceil \log_2 k \rceil$ for any positive integer $k \geq 2$. The Gray codes for the even values of k are closed, and the Gray codes for odd values of k are open.*

Gray Codes (cont.)

- 00, 01, 11 (open Gray code for 3 objects)
- 000, 001, 011 (add a 0)
- 100, 101, 111 (add a 1)
- Combine the preceding two codes (read the second in reversed order): 000, 001, 011, 111, 101, 100 (closed Gray code for 6 objects)

Gray Codes (cont.)

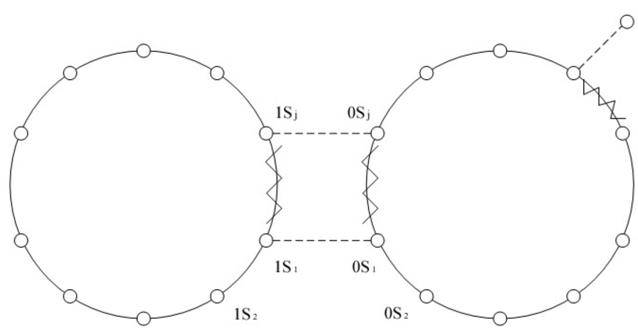


Figure 2.5 Constructing an open Gray code

Source: [Manber 1989] (adapted).

4 Reversed Induction

Arithmetic vs. Geometric Mean

Theorem 20 (2.13)**.** *If $x_1, x_2, \ldots, x_n$ are all positive numbers, then* $(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

First use the standard induction to prove the case of powers of 2 and then use the reversed induction principle below to prove for all natural numbers.

Theorem 21 (Reversed Induction Principle)**.** *If a statement P, with a parameter n, is true for an infinite subset of the natural numbers, and if, for every $n > 1$, the truth of P for n implies its truth for $n-1$, then P is true for all natural numbers.*

Arithmetic vs. Geometric Mean (cont.)

- For all powers of 2, i.e., $n = 2^k$, $k \geq 1$: by induction on k.
- Base case: $(x_1 x_2)^{\frac{1}{2}} \leq \frac{x_1+x_2}{2}$, squaring both sides ….
- Inductive step:

$$
\begin{aligned}
& (x_1 x_2 \cdots x_{2^{k+1}})^{\frac{1}{2^{k+1}}} \\
= \; & [(x_1 x_2 \cdots x_{2^{k+1}})^{\frac{1}{2^k}}]^{\frac{1}{2}} \\
= \; & [(x_1 x_2 \cdots x_{2^k})^{\frac{1}{2^k}} (x_{2^k+1} x_{2^k+2} \cdots x_{2^{k+1}})^{\frac{1}{2^k}}]^{\frac{1}{2}} \\
\leq \; & \frac{(x_1 x_2 \cdots x_{2^k})^{\frac{1}{2^k}} + (x_{2^k+1} x_{2^k+2} \cdots x_{2^{k+1}})^{\frac{1}{2^k}}}{2}, \text{ from the base case} \\
\leq \; & \frac{\frac{x_1+x_2+\cdots+x_{2^k}}{2^k} + \frac{x_{2^k+1}+x_{2^k+2}+\cdots+x_{2^{k+1}}}{2^k}}{2}, \text{ from the Ind. Hypo.} \\
= \; & \frac{x_1+x_2+\cdots+x_{2^{k+1}}}{2^{k+1}}
\end{aligned}
$$

Arithmetic vs. Geometric Mean (cont.)

- For all natural numbers: by reversed induction on n.
- Base case: the theorem holds for all powers of 2.
- Inductive step: observe that

$$
\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} = \frac{x_1 + x_2 + \cdots + x_{n-1} + \frac{x_1+x_2+\cdots+x_{n-1}}{n-1}}{n}.
$$

Arithmetic vs. Geometric Mean (cont.)

$$
\begin{aligned}
(x_1 x_2 \cdots x_{n-1} (\tfrac{x_1+x_2+\cdots+x_{n-1}}{n-1}))^{\frac{1}{n}} & \leq \frac{x_1+x_2+\cdots+x_{n-1}+\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}}{n} \\
& \quad \text{(from the Ind. Hypo.)} \\
(x_1 x_2 \cdots x_{n-1} (\tfrac{x_1+x_2+\cdots+x_{n-1}}{n-1}))^{\frac{1}{n}} & \leq \frac{x_1+x_2+\cdots+x_{n-1}}{n-1} \\
(x_1 x_2 \cdots x_{n-1} (\tfrac{x_1+x_2+\cdots+x_{n-1}}{n-1})) & \leq (\tfrac{x_1+x_2+\cdots+x_{n-1}}{n-1})^n \\
(x_1 x_2 \cdots x_{n-1}) & \leq (\tfrac{x_1+x_2+\cdots+x_{n-1}}{n-1})^{n-1} \\
(x_1 x_2 \cdots x_{n-1})^{\frac{1}{n-1}} & \leq (\tfrac{x_1+x_2+\cdots+x_{n-1}}{n-1})
\end{aligned}
$$

5 Loop Invariants

Loop Invariants

- An *invariant* at some point of a program is an assertion that holds whenever execution of the program reaches that point.
- Invariants are a bridge between the static text of a program and its dynamic computation.
- An invariant at the front of a while loop is called a *loop invariant* of the while loop.
- A loop invariant is formally established by induction.
  - Base case: the assertion holds right before the loop starts.
  - Inductive step: assuming the assertion holds before the i-th iteration ($i \geq 1$), it holds again after the iteration.

Number Conversion

```
Algorithm Convert_to_Binary(n);
begin
    t := n;
    k := 0;
    while t > 0 do
        k := k + 1;
        b[k] := t mod 2;
        t := t div 2;
end
```

Number Conversion (cont.)

Theorem 22 (2.14)**.** *When Algorithm Convert_to_Binary terminates, the binary representation of n is stored in the array b.*

Lemma 23. *If m is the integer represented by the binary array $b[1..k]$, then $n = t \cdot 2^k + m$ is a loop invariant of the while loop.*

See separate handout for a detailed proof.