Homework Assignment #2

Note

This assignment is due 2:10PM Tuesday, March 20, 2018. Please write or type your answers on A4 (or similar size) paper. Drop your homework by the due time in Yih-Kuen Tsay's mail box on the first floor of Management College Building 2. Late submission will be penalized by 20% for each working day overdue. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

There are five problems in this assignment, each accounting for 20 points. You must use *induction* for all proofs. (Note: problems marked with "(X.XX)" are taken from [Manber 1989] with probable adaptation.)

1. Reprove the following theorem which we have proven (mostly) in class. This time you must apply the *reversed induction* principle, or a variant of it, in some part of the proof. You may reuse some of the results that we have obtained in class without giving detailed proofs. Your main task is to demonstrate the use of reversed induction.

   > There exist Gray codes of length $\lceil \log_2 k \rceil$ for any positive integer $k \geq 2$. The Gray codes for the *even* values of k are *closed*, and the Gray codes for *odd* values of k are *open*.

2. Consider the following recurrence relation:

$$\begin{cases} T(0) = 0 \\ T(1) = 1 \\ T(h) = T(h-1) + T(h-2) + 1, \quad h \geq 2 \end{cases}$$

   Prove by induction the relation $T(h) = F(h+2) - 1$, where $F(n)$ is the n-th Fibonacci number ($F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$, for $n \geq 3$).

3. (2.30) A **full binary tree** is defined inductively as follows. A full binary tree of height 0 consists of 1 node which is the root. A full binary tree of height $h+1$ consists of two full binary trees of height h whose roots are connected to a new root. Let T be a full binary tree of height h. The **height** of a node in T is h minus the node's distance from the root (e.g., the root has height h, whereas a leaf has height 0). Prove that the sum of the heights of all the nodes in T is $2^{h+1} - h - 2$.

4. (2.32) Let n and m be integers such that $1 \leq m \leq n$. Prove by induction that

$$n^2 - m(n+1) + 2n + m^2 \leq n^2 + n$$

   (Hint: Use a "two sided" induction on m. Prove two base cases, $m = 1$ and $m = n$, and go either forward from $m = 1$ or backward from $m = n$.)

5. Consider again the inductive definition in Homework Assignment #1 for the set of all binary trees that store non-negative integer key values:

(a) The empty tree, denoted $\perp$, is a binary tree.

(b) If t_l and t_r are binary trees, then $node(k, t_l, t_r)$, where $k \in Z$ and $k \geq 0$, is also a binary tree.

Refine the definition to include only binary search trees such that an inorder traversal of a binary search tree produces a list of all stored key values in *increasing* order. Then, define a function that outputs the rank of a given key value (the position of the key value in the aforementioned sorted list) if it is stored in the tree, or 0 if the key is not in the tree.