Homework 3

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Question1

進行代換 $[f(n) \mapsto log_2(n), c \mapsto a, a \mapsto 2^b]$
小心這兩個 a 是不一樣的 a，容易腦袋打結

Question2(a)

f(n)= $(\log n)^{\log n}$,g(n)= $\frac{n}{\log n}$

Claim f(n)=Ω(g(n)), $\exists$ c $\exists$ N $\forall$ n $\geq$N, let $c = 1$

$(logn)^{logn}$$\geq$c $\cdot$ $(\frac{n}{logn})$

$\xLeftarrow{logn \text{ 以 } x \text{ 代入}}$ $x^x \geq$ c$(\frac{2^x}{x})$

$\xLeftarrow{\text{兩邊同取 } log_2}$ $xlogx \geq x + logc - logx$ (當 x>1 成立)

When n>2,c=1

f(n)=Ω(g(n))

Question2(a)

To prove $f(n) \neq O(g(n))$, we prove $f(n) = \omega(g(n))$, that is, $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{\frac{n}{logn}}{logn^{logn}} \\
&= \lim_{n\to\infty} \frac{n}{logn \cdot logn^{logn}} \\
&= \lim_{n\to\infty} \frac{n}{logn^{(logn)+1}} \\
&= \lim_{n\to\infty} \frac{1}{logn \cdot \frac{1}{n} loge} \text{ (By l'Hôpital's rule)} \\
&= 0
\end{aligned}
$$

Hence $f(n) = \omega(g(n)) \neq O(g(n))$.

Question2(b)

$f(n) = n^2 2^n, g(n) = 3^n$

Guess $f(n) = O(g(n))$, then there exist constants c and N such that, for all $n \geq N$, $f(n) \leq cg(n)$.Let $c = 1$
$n^2 2^n \leq c3^n$
$\Leftarrow n^2 \leq c(\frac{3}{2})^n$
$\Leftarrow 2\log n \leq n(\log c + \log 3 - \log 2)$
$\Leftarrow 2\log n \leq 0.586n$
When $n = 13$, $2\log 13 \approx 7.4 \leq 0.586 \times 13 \approx 7.6$
We find when $c = 1$, $n \geq 13$, $n^2 2^n \leq c3^n$.
Hence $f(n) = O(g(n))$.

Question2(b) cont'd

To prove $f(n) \neq \Omega(g(n))$, we prove $f(n) = o(g(n))$, that is, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n^2 2^n}{3^n} \\
&= \lim_{n\to\infty} \frac{n^2}{(\frac{3}{2})^n} \\
&= \lim_{n\to\infty} \frac{2n}{(\ln \frac{3}{2})(\frac{3}{2})^n} \text{ (By l'Hôpital's rule)} \\
&= \lim_{n\to\infty} \frac{2}{(\ln \frac{3}{2})^2(\frac{3}{2})^n} \text{ (By l'Hôpital's rule)} \\
&= 0
\end{aligned}$$

Hence $f(n) = o(g(n)) \neq \Omega(g(n))$.

Question3

$$T(1)=1$$

$$T(2)=2+T(1)$$

$$T(3)=3+T(2)+T(1)$$

$$\vdots$$

$$T(n\text{-}1)=(n\text{-}1)+[T(n\text{-}2)+T(n\text{-}3)+\ldots] \ldots \text{①}$$

$$T(n)=(n)+[T(n\text{-}1)+T(n\text{-}2)+T(n\text{-}3)+\ldots]\ldots \text{②}$$

Question3

$T(n\text{-}1)=(n\text{-}1)+[T(n\text{-}2)+T(n\text{-}3)+\ldots]\ \ldots$ ①

$T(n)=(n)+[T(n\text{-}1)+T(n\text{-}2)+T(n\text{-}3)+\ldots]\ldots$ ②

② $-$ ① :

$T(n)\text{-}T(n\text{-}1)=[n\text{-}(n\text{-}1)]+T(n\text{-}1)$

$T(n)=2T(n\text{-}1)+1$

Question3

$$\begin{aligned}
T(n) &= 2T(n\text{-}1)+1 \\
&= 2[2T(n\text{-}2)+1]+1 \\
&= 2^2T(n\text{-}2)+(2+1) \\
&= 2^2[2T(n\text{-}3)+1]+(2+1) \\
&= 2^3T(n\text{-}3)+(2^2+2+1) \\
&\vdots \\
&= 2^iT(n\text{-}i)+(1+2+2^2+\ldots+2^{i-1})
\end{aligned}$$

Question3

2^iT(n-i)+$(1+2+2^2+\ldots+2^{i-1})$

i 以 (n-1) 代入:

$=2^{n-1}\cdot$T(1)+$(1+2+\ldots+2^{n-2})$

$=2^{n-1}\cdot 1+\frac{2(2^{n-1}-1)}{2}$

$=2^n$-1

Question4

Consider the recurrence relation

$$T(n) = 2T(n/2) + 1, T(2) = 1.$$

We try to prove that $T(n) = O(n)$ (we limit our attention to powers of 2). We guess that $T(n) \leq cn$ for some (as yet unknown) c, and substitute cn in the expression. We have to show that $cn \geq 2c(n/2) + 1$. But this is clearly not true. Find the correct solution of this recurrence (you can assume that n is a power of 2), and explain why this attempt failed.

Question4(cont'd)

The attempt in this question failed because we have no way to eliminate the positive constant(1 in this case), which would accumulate during the recursion.
We may try a more strict guess: $T(n) \leq cn - 1$, which implies $T(n/2) \leq cn/2 - 1$.
If we substituting the upper bound $cn/2 - 1$ for $T(n/2)$ in the induction step, we get

$$\begin{aligned} T(n) &= 2T(n/2) + 1 \\ &\leq 2(cn/2 - 1) + 1 \\ &= cn - 2 + 1 \\ &= cn - 1 \\ &\leq cn \end{aligned}$$

Hence we have proven that $T(n) \leq cn$, implying $T(n) = O(n)$.

Question5

使用母函數（生成函數）

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

由於題目的數列編號從 1 開始，也可以讓 n 從 1 開始
以下解答使用 $n = 0$ 的版本

Question5

我們有 $T_n = T_{n-1} + 2T_{n-2}$

$$
\begin{array}{lccccccccc}
 & F(x) & = & T_1 & + & T_2 & x & + & T_3 & x^2 & \dots \\
-x & F(x) & = & & - & T_1 & x & - & T_2 & x^2 & \dots \\
-2x^2 & F(x) & = & & & & & - & T_1 & x^2 & \dots \\
\hline
(1-x-2x^2) & F(x) & = & T_1 & + & (T_2 - T_1) & x & & & &
\end{array}
$$

$$
\begin{aligned}
F(x) &= \frac{1+x}{1-x-2x^2} \\
&= \frac{1+x}{(1+x)(1-2x)} \\
&= \frac{1}{1-2x} \\
&= \sum_{n=0}^{\infty} 2^n x^n
\end{aligned}
$$

Question5

$$\begin{aligned} F(x) &= T_1 + T_2\, x + T_3\, x^2 + \ldots \\ &= 1 + 2\, x + 4\, x^2 + \ldots + 2^n x^n + \ldots \end{aligned}$$

很明顯地，T_1 對應到 1，T_2 對應到 2，
而 T_n 對應到 x^{n-1} 的係數：2^{n-1}

Question5

常錯的點！
看到 $\sum_{n=0}^{\infty} \mathbf{2^n} x^n$ 就寫下 $T(n) = 2^n$
無論 n 從 0 還是 1 開始，概念都是：**母函數的寫法要前後一致**

若一開始把 $T(1)$ 和 x^0 放在一起，最後你會得到 $\sum_{n=0}^{\infty} 2^n x^n$
與 x^0 在一起的就會是 1
T_n 對應到 x^{n-1}，係數為 2^{n-1}

若一開始 $T(1)$ 是和 x^1 放在一起，那應該會得到 $\sum_{n=1}^{\infty} 2^{n-1} x^n$
與 x^1 在一起的也會是 1
T_n 對應到 x^n，係數也會是 2^{n-1}

較好的作法是後者，可想像成把 $T_0 = 0$ 塞進去，比較不會出錯