

# Algorithms 2021: Reduction

(Based on [Manber 1989])

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## 1 Introduction

### Introduction

- The basic idea of *reduction* is to solve a problem with the solution to another “similar” problem.
- When Problem  $A$  can be reduced (efficiently) to Problem  $B$ , there are two consequences:
  - A solution to Problem  $B$  may be used to solve Problem  $A$ .
  - If  $A$  is known to be “hard”, then  $B$  is also necessarily “hard”.

/\* A reduction should be reasonably efficient (this will be made precise in the topic of NP-completeness). Otherwise, one might be able to reduce a hard problem to a simpler one, by solving the more time-consuming part during the process of reduction and leaving the easier part to the second problem. \*/

- One should avoid the pitfall of reducing a problem to another that is too general or too hard.

## 2 Bipartite Matching

### Matching

- Given an undirected graph  $G = (V, E)$ , a **matching** is a set of edges that do not share a common vertex.
- A **maximum** matching is one with the maximum number of edges.
- A **maximal** matching is one that cannot be extended by adding any other edge.

### Bipartite Matching

- A bipartite graph  $G = (V, E, U)$  is a graph with  $V \cup U$  as the set of vertices and  $E$  as the set of edges such that
  - $V$  and  $U$  are disjoint and
  - The edges in  $E$  connect vertices from  $V$  to vertices in  $U$ .

**Problem 1.** *Given a bipartite graph  $G = (V, E, U)$ , find a maximum matching in  $G$ .*

### Bipartite Matching (cont.)

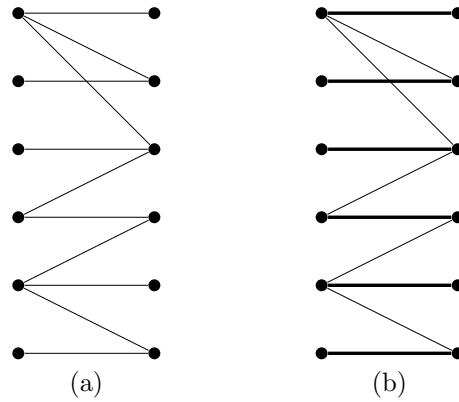


Figure: A bipartite graph and a maximum matching.

Source: adapted from [Manber 1989, Figure 7.37].

### Bipartite Matching (cont.)

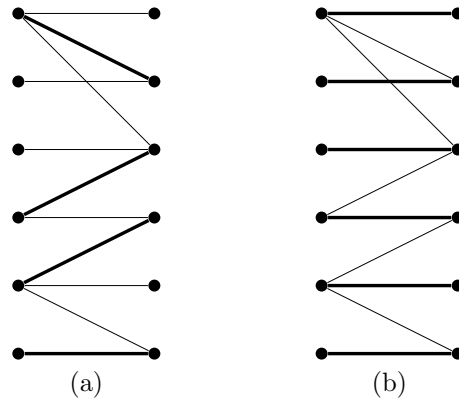


Figure: A maximal matching and a maximum matching.

Source: adapted from [Manber 1989, Figure 7.37].

## 3 Network Flows

### Networks

- Consider a directed graph, or network,  $G = (V, E)$  with two distinguished vertices:  $s$  (the source) with indegree 0 and  $t$  (the sink) with outdegree 0.
- Each edge  $e$  in  $E$  has an associated positive weight  $c(e)$ , called the *capacity* of  $e$ .

### The Network Flow Problem

- A **flow** is a function  $f$  on  $E$  that satisfies the following two conditions:
  1.  $0 \leq f(e) \leq c(e)$ .
  2.  $\sum_u f(u, v) = \sum_w f(v, w)$ , for all  $v \in V - \{s, t\}$ .
- The **network flow problem** is to maximize the flow  $f$  for a given network  $G$ .

## 4 Bipartite Matching to Network Flow

### Bipartite Matching to Network Flow

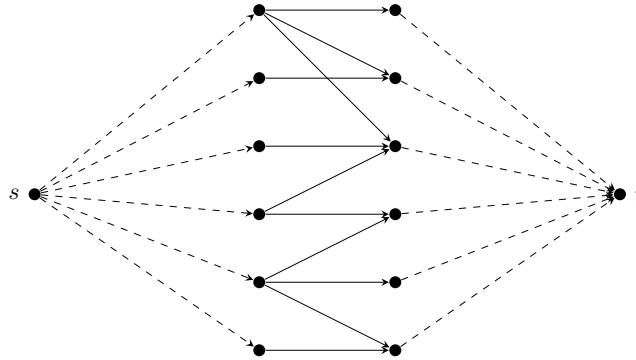


Figure: Reducing bipartite matching to network flow. Every edge has capacity 1.

Source: redrawn from [Manber 1989, Figure 7.39].

### Bipartite Matching to Network Flow (cont.)

- Mapping from the input  $G = (V, E, U)$  of the bipartite matching problem to the input  $G' = (V', E')$  and  $c$  of the network flow problem:
  - The network is  $G' = (V', E')$  where
    - \*  $V' = \{s\} \cup V \cup U \cup \{t\}$
    - \*  $E' = \{(s, v) \mid v \in V\} \cup E \cup \{(u, t) \mid u \in U\}$
  - The capacity for every  $e \in E'$  is 1, i.e.,  $\forall e \in E', c(e) = 1$ .
- Correspondence between the two solutions
  - A maximum flow  $f$  in  $G'$  defines a maximum matching  $M_f$  in  $G$ .
  - A maximum matching  $M$  in  $G$  induces a maximum flow  $f_M$  in  $G'$ .

## 5 Linear Programming

### Notations

- Let  $\bar{v}$  denote a vector  $(v_1, v_2, \dots, v_n)$  of  $n$  constants or  $n$  variables.
- In the following,  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$ , and  $\bar{e}$  are vectors of  $n$  constants.
- And,  $\bar{x}$  and  $\bar{y}$  are vectors of  $n$  variables.
- The (inner or dot) product  $\bar{a} \cdot \bar{x}$  of two vectors  $\bar{a}$  and  $\bar{x}$  is defined as follows:

$$\bar{a} \cdot \bar{x} = \sum_{i=1}^n a_i \cdot x_i$$

## Linear Programming

- Objective function:

$$\bar{c} \cdot \bar{x}$$

- Equality constraints:

$$\begin{aligned}\bar{e}_1 \cdot \bar{x} &= d_1 \\ \bar{e}_2 \cdot \bar{x} &= d_2 \\ &\vdots \\ \bar{e}_m \cdot \bar{x} &= d_m\end{aligned}$$

- Inequality constraints may be turned into equality constraints by introducing *slack* variables.
- Non-negative constraints:  $x_j \geq 0$ , for all  $j$  in  $P$ , where  $P$  is a subset of  $\{1, 2, \dots, n\}$ .
- The goal is to *maximize* (or *minimize*) the value of the objective function, subject to the equality constraints.

## 6 Network Flow to Linear Programming

### Network Flow to Linear Programming

- From the input  $G = (V, E)$  and  $c$  of the network flow problem to the objective function and constraints of linear programming:

- Let  $x_1, x_2, \dots, x_n$  represent the flow values of the  $n$  edges.
- Objective function:

$$\sum_{i \in S} x_i$$

where  $S$  is the set of edges leaving the source.

- Inequality constraints:

$$x_i \leq c_i, \text{ for all } i, 1 \leq i \leq n$$

where  $c_i$  is the capacity of edge  $i$ .

- Equality constraints:

$$\sum_{i \text{ leaves } v} x_i - \sum_{j \text{ enters } v} x_j = 0, \text{ for every } v \in V \setminus \{s, t\}$$

- Non-negative constraints:  $x_i \geq 0$ , for all  $i, 1 \leq i \leq n$ .