String Processing

(Based on [Manber 1989])

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Data Compression

Problem

Given a text (a sequence of characters), find an encoding for the characters that satisfies the prefix constraint and that minimizes the total number of bits needed to encode the text.

The *prefix constraint* states that the prefixes of an encoding of one character must not be equal to a complete encoding of another character.

Denote the characters by $c_1, c_2, \cdots, c_n$ and their frequencies by $f_1, f_2, \cdots, f_n$. Given an encoding E in which a bit string s_i represents c_i, the length (number of bits) of the text encoded by using E is $\sum_{i=1}^{n} |s_i| \cdot f_i$.

A Code Tree

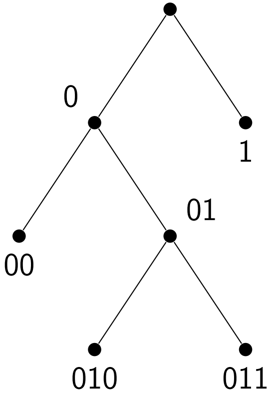

Figure: The tree representation of encoding.

Source: redrawn from [Manber 1989, Figure 6.17].

A Huffman Tree

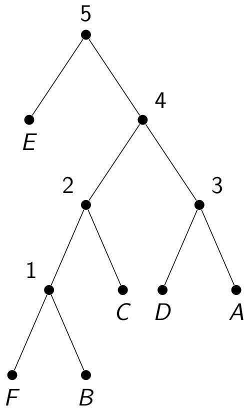

Figure: The Huffman tree for a text with frequencies of A: 5, B: 2, C: 3, D: 4, E: 10, F:1. The code of B, for example, is 1001. The numbers labeling the internal nodes indicate the order in which the corresponding subtrees are formed.

Source: redrawn from [Manber 1989, Figure 6.19].

Huffman Encoding

```
Algorithm Huffman_Encoding ($S, f$);
    insert all characters into a heap $H$
        according to their frequencies;
    while $H$ not empty do
        if $H$ contains only one character $X$ then
            make $X$ the root of $T$
        else
            delete $X$ and $Y$ with lowest frequencies;
                from $H$;
            create $Z$ with a frequency equal to the
                sum of the frequencies of $X$ and $Y$;
            insert $Z$ into $H$;
            make $X$ and $Y$ children of $Z$ in $T$
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What is its time complexity? $O(n \log n)$

String Matching

Problem

Given two strings A $(= a_1 a_2 \cdots a_n)$ *and* B $(= b_1 b_2 \cdots b_m)$, *find the first occurrence (if any) of* B *in* A. *In other words, find the smallest* k *such that, for all* i, $1 \leq i \leq m$, *we have* $a_{k-1+i} = b_i$.

A (non-empty) *substring* of a string A is a consecutive sequence of characters $a_i a_{i+1} \cdots a_j$ $(i \leq j)$ from A.

Straightforward String Matching

$A = xyxxyxyxyyxyxyxyyxyxyxx. \quad B = xyxyyxyxyxx.$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	x	y	x	x	y	x	y	x	y	y	x	y	x	y	x	y	y	x	y	x	y	x	x
1 :	x	y	x	y	·	·	·																
2 :		x	·	·	·																		
3 :			x	y	·	·	·																
4 :				x	y	x	y	y	·	·	·												
5 :					x	·	·	·															
6 :						x	y	x	y	y	x	y	x	y	x	x							
7 :							x	·	·	·													
8 :								x	y	x	·	·	·										
9 :									x	·	·	·											
10 :										x	·	·	·										
11 :											x	y	x	y	y	·	·	·					
12 :												x	·	·	·								
13 :													x	y	x	y	y	x	y	x	y	x	x

Figure: An example of a straightforward string matching.

Source: redrawn from [Manber 1989, Figure 6.20].

Straightforward String Matching (cont.)

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  - B $(= b_1 b_2 \cdots b_m)$ may be compared against
    - $a_1 a_2 \cdots a_m$,
    - $a_2 a_3 \cdots a_{m+1}$,
    - $\ldots$, and
    - $a_{n-m+1} a_{n-m+2} \cdots a_n$
  - For example, $A = xxxx \ldots xxxy$ and $B = xxxy$.

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  - For example, $A = xxxx \ldots xxxy$ and $B = xxxy$.
- So, the time complexity is $O(m \times n)$.
- But the best possible is linear-time, with a preprocessing.
- The cause of deficiency: tries from 7 to 12 in the example are doomed to fail. Why?
- How can we avoid the futile tries?

Matching the Pattern Against Itself

- In the example, when the ongoing matching fails at b_{11} against a_{16}, we know that $b_1 b_2 \ldots b_{10}$ equals $a_6 a_7 \ldots a_{15}$.

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- We can tell this by just looking at B, as $a_{13} a_{14} a_{15}$ equals $b_8 b_9 b_{10}$.

$$\begin{array}{cccccccccccc}
B = & x & y & x & y & y & x & y & x & y & x & x \\
 & & x & \cdot & \cdot & \cdot & & & & & & \\
 & & & x & y & x & \cdot & \cdot & \cdot & & & \\
 & & & & x & \cdot & \cdot & \cdot & & & & \\
 & & & & & x & \cdot & \cdot & \cdot & & & \\
 & & & & & & x & y & x & y & y & \\
 & & & & & & & x & \cdot & \cdot & \cdot & \\
 & & & & & & & & x & y & x &
\end{array}$$

Figure: Matching the pattern against itself.

Source: redrawn from [Manber 1989, Figure 6.21].

The Values of *next*

$i =$	1	2	3	4	5	6	7	8	9	10	11
$B =$	x	y	x	y	y	x	y	x	y	x	x
$next =$	-1	0	0	1	2	0	1	2	3	4	3

Figure: The values of *next*.

Source: redrawn from [Manber 1989, Figure 6.22].

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The value of $next[j]$ tells the length of the longest proper prefix that is equal to a suffix of $b_1 b_2 \ldots b_{j-1}$.

If the ongoing matching fails at b_j against a_i, then $b_{next[j]+1}$ is the next to try against a_i.

Note: $next[1]$ is set to -1 so that this unique case is easily differentiated (see the main loop of the KMP algorithm).

The KMP Algorithm

```
Algorithm String_Match $(A, n, B, m)$;
begin
    $j := 1;\ \ i := 1;$
    $Start := 0;$
    while $Start = 0$ and $i \leq n$ do
        if $B[j] = A[i]$ then
            $j := j + 1;\ \ i := i + 1$
        else
            $j := next[j] + 1;$
            if $j = 0$ then
                $j := 1;\ \ i := i + 1;$
        if $j = m + 1$ then $Start := i - m$
end
```

The KMP Algorithm (cont.)

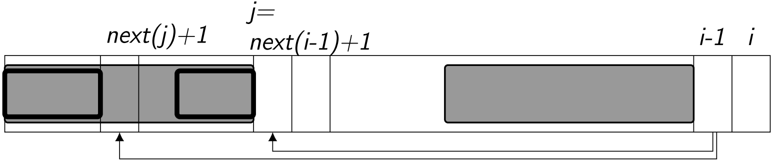

Figure: Computing *next*(i).

Source: redrawn from [Manber 1989, Figure 6.24].

The KMP Algorithm (cont.)

```
Algorithm Compute_Next ($B, m$);
begin
    $next[1] := -1$;  $next[2] := 0$;
    for $i := 3$ to $m$ do
        $j := next[i-1] + 1$;
        while $B[i-1] \neq B[j]$ and $j > 0$ do
            $j := next[j] + 1$;
        $next[i] := j$
end
```

The KMP Algorithm (cont.)

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  - However, for these to happen, each of $a_{i-j+2}, a_{i-j+3}, \ldots, a_{i-1}$ was compared against the corresponding character in $b_1 b_2 \ldots b_{j-1}$ just once.

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- Every a_i is incurred the cost of at most two comparisons.
- So, the time complexity is $O(n)$.

String Editing

Problem

Given two strings A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the minimum number of changes required to change A character by character such that it becomes equal to B.

Three types of changes (or edit steps) allowed: (1) insert, (2) delete, and (3) replace.

String Editing (cont.)

Let $C(i, j)$ denote the minimum cost of changing $A(i)$ to $B(j)$, where $A(i) = a_1 a_2 \cdots a_i$ and $B(j) = b_1 b_2 \cdots b_j$.

For $i = 0$ or $j = 0$,

$$\begin{aligned} C(i,0) &= i \\ C(0,j) &= j \end{aligned}$$

For $i > 0$ and $j > 0$,

$$C(i,j) = \min \begin{cases} C(i-1,j)+1 & (\text{deleting } a_i) \\ C(i,j-1)+1 & (\text{inserting } b_j) \\ C(i-1,j-1)+1 & (a_i \to b_j) \\ C(i-1,j-1) & (a_i = b_j) \end{cases}$$

String Editing (cont.)

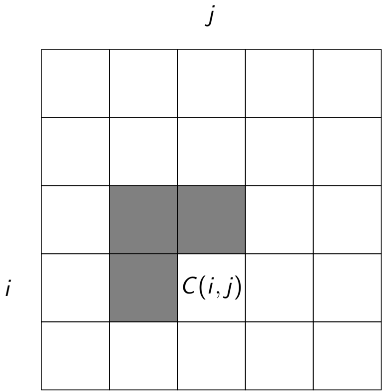

Figure: The dependencies of $C(i, j)$.

Source: redrawn from [Manber 1989, Figure 6.26].

String Editing (cont.)

```
Algorithm Minimum_Edit_Distance ($A, n, B, m$);
    for $i$ := 0 to $n$ do $C[i, 0] := i$;
    for $j$ := 1 to $m$ do $C[0, j] := j$;
    for $i$ := 1 to $n$ do
        for $j$ := 1 to $m$ do
            $x := C[i-1, j] + 1$;
            $y := C[i, j-1] + 1$;
            if $a_i = b_j$ then
                $z := C[i-1, j-1]$
            else
                $z := C[i-1, j-1] + 1$;
        $C[i, j] := min(x, y, z)$
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        **for** $j := 1$ **to** $m$ **do**
            $x := C[i-1, j] + 1$;
            $y := C[i, j-1] + 1$;
            **if** $a_i = b_j$ **then**
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Its time complexity is clearly $O(mn)$.