

Advanced Graph Algorithms

(Based on [Manber 1989])

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Strongly Connected Components

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Strongly Connected Components

- 🌐 A directed graph is *strongly connected* if there is a directed path from every vertex to every other vertex.
- 🌐 A *strongly connected component* (SCC) is a maximal subset of the vertices such that its induced subgraph is strongly connected (namely, there is no other subset that contains it and induces a strongly connected graph).

Strongly Connected Components (cont.)

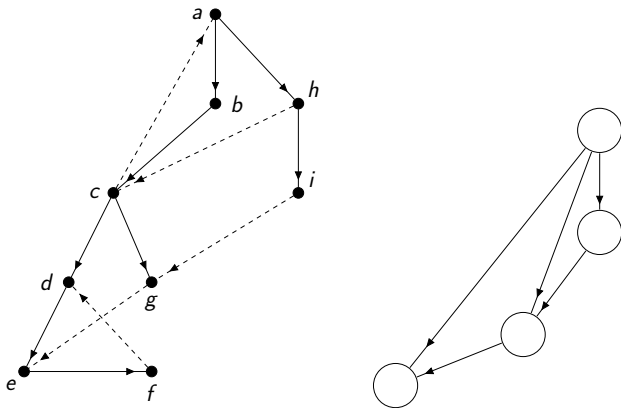


Figure: A directed graph and its strongly connected component graph.

Source: redrawn from [Manber 1989, Figure 7.30].

Lemma (7.11)

Two distinct vertices belong to the same SCC if and only if there is a circuit containing both of them.

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Two distinct vertices belong to the same SCC if and only if there is a circuit containing both of them.

Lemma (7.12)

Each vertex belongs to exactly one SCC.

Strongly Connected Components (cont.)

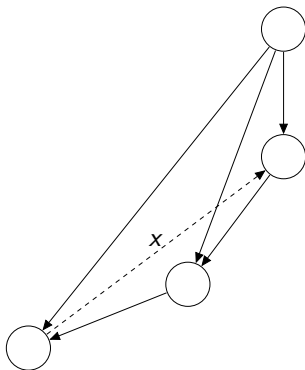


Figure: Adding an edge connecting two different strongly connected components.

Source: redrawn from [Manber 1989, Figure 7.31].

Strongly Connected Components (cont.)

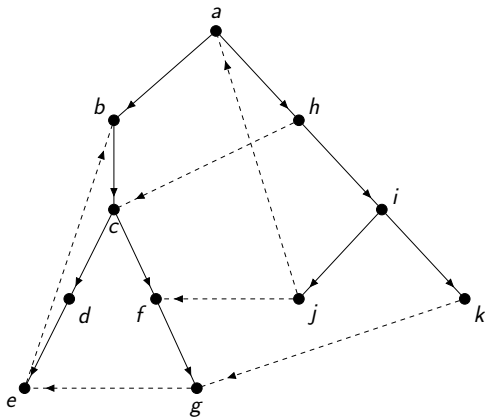


Figure: The effect of cross edges.

Source: redrawn from [Manber 1989, Figure 7.32].

Strongly Connected Components (cont.)

Algorithm Strongly_Connected_Components(G, n);

begin

for every vertex v of G **do**

$v.DFS_Number := 0$;

$v.Component := 0$;

$Current_Component := 0$; $DFS_N := n$;

while $v.DFS_Number = 0$ for some v **do**

$SCC(v)$

end

procedure $SCC(v)$;

begin

$v.DFS_Number := DFS_N$;

$DFS_N := DFS_N - 1$;

insert v into $Stack$;

$v.High := v.DFS_Number$;

Strongly Connected Components (cont.)

```
for all edges  $(v, w)$  do  
    if  $w.DFS\_Number = 0$  then  
         $SCC(w)$ ;  
         $v.High := \max(v.High, w.High)$   
    else if  $w.DFS\_Number > v.DFS\_Number$   
        and  $w.Component = 0$  then  
             $v.High := \max(v.High, w.DFS\_Number)$   
            //  $\max(v.High, w.High)$  also works  
if  $v.High = v.DFS\_Number$  then  
     $Current\_Component := Current\_Component + 1$ ;  
    repeat  
        remove  $x$  from the top of  $Stack$ ;  
         $x.component := Current\_Component$   
    until  $x = v$   
end
```

Strongly Connected Components (cont.)

```
for all edges  $(v, w)$  do  
    if  $w.DFS\_Number = 0$  then  
         $SCC(w)$ ;  
         $v.High := \max(v.High, w.High)$   
    else if  $w.DFS\_Number > v.DFS\_Number$   
        and  $w.Component = 0$  then  
             $v.High := \max(v.High, w.DFS\_Number)$   
            //  $\max(v.High, w.High)$  also works  
if  $v.High = v.DFS\_Number$  then  
     $Current\_Component := Current\_Component + 1$ ;  
    repeat  
        remove  $x$  from the top of  $Stack$ ;  
         $x.component := Current\_Component$   
    until  $x = v$   
end
```

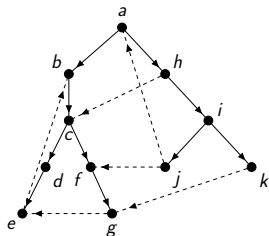
Time complexity:

Strongly Connected Components (cont.)

```
for all edges  $(v, w)$  do
  if  $w.DFS\_Number = 0$  then
     $SCC(w)$ ;
     $v.High := \max(v.High, w.High)$ 
  else if  $w.DFS\_Number > v.DFS\_Number$ 
    and  $w.Component = 0$  then
     $v.High := \max(v.High, w.DFS\_Number)$ 
    //  $\max(v.High, w.High)$  also works
  if  $v.High = v.DFS\_Number$  then
     $Current\_Component := Current\_Component + 1$ ;
    repeat
      remove  $x$  from the top of  $Stack$ ;
       $x.component := Current\_Component$ 
    until  $x = v$ 
end
```

Time complexity: $O(|E| + |V|)$.

Strongly Connected Components (cont.)



	a	b	c	d	e	f	g	h	i	j	k
	11	10	9	8	7	6	5	4	3	2	1
a	11	-	-	-	-	-	-	-	-	-	-
b	11	10	-	-	-	-	-	-	-	-	-
c	11	10	9	-	-	-	-	-	-	-	-
d	11	10	9	8	-	-	-	-	-	-	-
e	11	10	9	8	10	-	-	-	-	-	-
d	11	10	9	10	10	-	-	-	-	-	-
c	11	10	10	10	10	-	-	-	-	-	-
f	11	10	10	10	10	6	-	-	-	-	-
g	11	10	10	10	10	6	7	-	-	-	-
f	11	10	10	10	10	7	7	-	-	-	-
c	11	10	10	10	10	7	7	-	-	-	-
(b)	11	10	10	10	10	7	7	-	-	-	-
a	11	10	10	10	10	7	7	-	-	-	-
h	11	10	10	10	10	7	7	4	-	-	-
i	11	10	10	10	10	7	7	4	3	-	-
j	11	10	10	10	10	7	7	4	3	11	-
i	11	10	10	10	10	7	7	4	11	11	-
(k)	11	10	10	10	10	7	7	4	11	11	1
i	11	10	10	10	10	7	7	4	11	11	1
h	11	10	10	10	10	7	7	11	11	11	1
(a)	11	10	10	10	10	7	7	11	11	11	1

Figure: An example of computing *High* values and strongly connected components.

Source: redrawn from [Manber 1989, Figure 7.34].

Odd-Length Cycles

Problem

Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle of odd length.


Odd-Length Cycles

Problem

Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle of odd length.

- 🌐 A cycle must reside completely within a strongly connected component (SCC), so we exam each SCC separately.
- 🌐 Mark the nodes of an SCC with “even” or “odd” using DFS.
- 🌐 If we have to mark a node that is already marked in the opposite, then we have found an odd-length cycle.

Biconnected Components

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Biconnected Components

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- 🌐 A graph is *not* biconnected if and only if there is a vertex whose removal disconnects the graph. Such a vertex is called an *articulation point*.

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- 🌐 A graph is *not* biconnected if and only if there is a vertex whose removal disconnects the graph. Such a vertex is called an *articulation point*.
- 🌐 A *biconnected component* (BCC) is a *maximal* subset of the edges such that its induced subgraph is biconnected (namely, there is no other subset that contains it and induces a biconnected graph).

Biconnected Components (cont.)

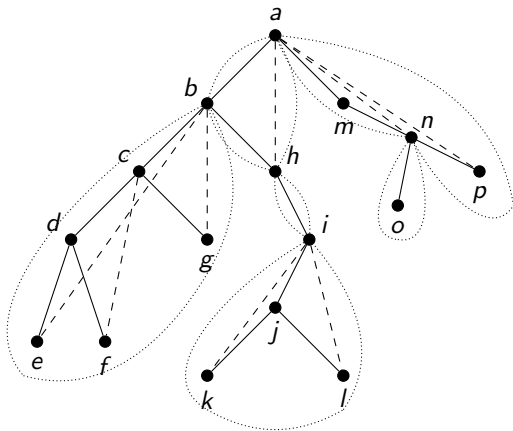


Figure: The structure of a nonbiconnected graph.

Source: redrawn from [Manber 1989, Figure 7.25].

Lemma (7.9)

Two distinct edges e and f belong to the same BCC if and only if there is a cycle containing both of them.

Biconnected Components (cont.)

Lemma (7.9)

Two distinct edges e and f belong to the same BCC if and only if there is a cycle containing both of them.

Lemma (7.10)

Each edge belongs to exactly one BCC.

Biconnected Components (cont.)

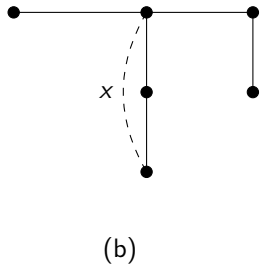
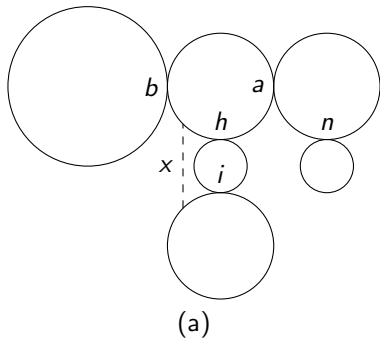


Figure: An edge that connects two different biconnected components. (a) The components corresponding to the graph of Figure 7.25 with the articulation points indicated. (b) The biconnected component tree.

Source: redrawn from [Manber 1989, Figure 7.26].

Biconnected Components (cont.)

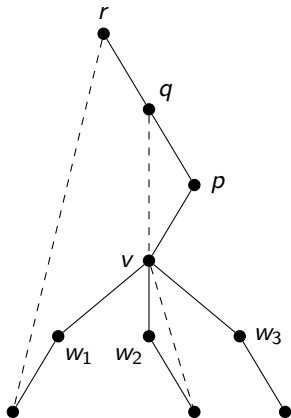


Figure: Computing the *High* values.

Source: redrawn from [Manber 1989, Figure 7.27].

Biconnected Components (cont.)

```
Algorithm Biconnected_Components( $G, v, n$ );  
begin  
  for every vertex  $w$  do  $w.DFS\_Number := 0$ ;  
   $DFS\_N := n$ ;  
   $BC(v)$   
end
```

```
procedure  $BC(v)$ ;  
begin  
   $v.DFS\_Number := DFS\_N$ ;  
   $DFS\_N := DFS\_N - 1$ ;  
  insert  $v$  into  $Stack$ ;  
   $v.High := v.DFS\_Number$ ;
```


Biconnected Components (cont.)

```
for all edges  $(v, w)$  do  
    insert  $(v, w)$  into Stack;  
    if  $w$  is not the parent of  $v$  then  
        if  $w.DFS\_Number = 0$  then  
             $BC(w)$ ;  
        if  $w.High \leq v.DFS\_Number$  then  
            remove all edges and vertices  
            from Stack until  $v$  is reached;  
            insert  $v$  back into Stack;  
             $v.High := \max(v.High, w.High)$   
        else  
             $v.High := \max(v.High, w.DFS\_Number)$   
            //  $\max(v.High, w.High)$  would not work, unlike in SCC  
    end
```

Biconnected Components (cont.)

procedure $BC(v)$;

begin

$v.DFS_Number := DFS_N$;

$DFS_N := DFS_N - 1$;

$v.High := v.DFS_Number$;

for all edges (v, w) **do**

if w is not the parent of v **then**

insert (v, w) into *Stack*;

if $w.DFS_Number = 0$ **then**

$BC(w)$;

if $w.high \leq v.DFS_Number$ **then**

remove all edges from *Stack*

until (v, w) is reached;

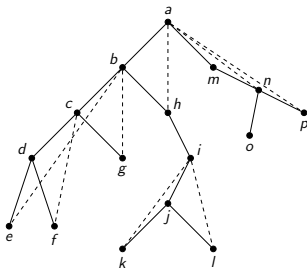
$v.High := \max(v.High, w.High)$

else

$v.High := \max(v.High, w.DFS_Number)$

end

Biconnected Components (cont.)



	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
a	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
b	16	15	-	-	-	-	-	-	-	-	-	-	-	-	-	-
c	16	15	14	-	-	-	-	-	-	-	-	-	-	-	-	-
d	16	15	14	13	-	-	-	-	-	-	-	-	-	-	-	-
e	16	15	14	13	15	-	-	-	-	-	-	-	-	-	-	-
f	16	15	14	15	15	-	-	-	-	-	-	-	-	-	-	-
g	16	15	14	15	15	14	-	-	-	-	-	-	-	-	-	-
h	16	15	15	15	15	14	15	-	-	-	-	-	-	-	-	-
i	16	15	15	15	15	14	15	16	8	-	-	-	-	-	-	-
j	16	15	15	15	15	14	15	16	8	7	-	-	-	-	-	-
k	16	15	15	15	15	14	15	16	8	7	8	-	-	-	-	-
l	16	15	15	15	15	14	15	16	8	8	8	-	-	-	-	-
m	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
n	16	16	15	15	15	14	15	16	8	8	8	8	4	-	-	-
o	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	-
p	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
a	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
b	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
c	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
d	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
e	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
f	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
g	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
h	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
i	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
j	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
k	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
l	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
m	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
n	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
o	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
p	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16

Figure: An example of computing the *High* values and biconnected components.

Source: redrawn from [Manber 1989, Figure 7.29].

Even-Length Cycles

Problem

Given a connected undirected graph $G = (V, E)$, determine whether it contains a cycle of even length.

Even-Length Cycles

Problem

Given a connected undirected graph $G = (V, E)$, determine whether it contains a cycle of even length.

Theorem

Every biconnected graph that has more than one edge and is not merely an odd-length cycle contains an even-length cycle.

Even-Length Cycles (cont.)

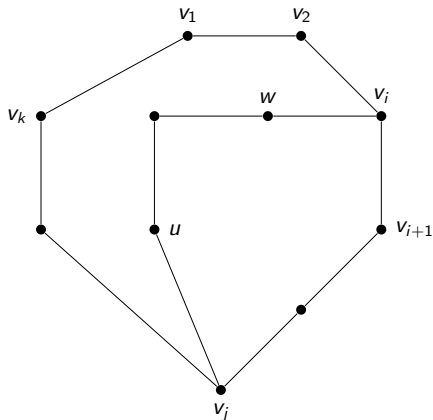


Figure: Finding an even-length cycle.

Source: redrawn from [Manber 1989, Figure 7.35].

Network Flows

- Consider a directed graph, or network, $G = (V, E)$ with two distinguished vertices: s (the **source**) with indegree 0 and t (the **sink**) with outdegree 0.
- Each edge e in E has an associated positive weight $c(e)$, called the *capacity* of e .

Network Flows (cont.)

🌐 A **flow** is a function f on E that satisfies the following two conditions:

1. $0 \leq f(e) \leq c(e)$.

2. $\sum_u f(u, v) = \sum_w f(v, w)$, for all $v \in V - \{s, t\}$.

🌐 The **network flow problem** is to maximize the flow f for a given network G .

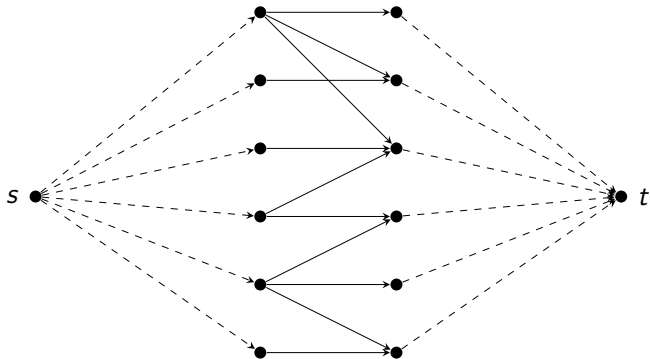




Figure: Reducing bipartite matching to network flow. Every edge has capacity 1.

Source: redrawn from [Manber 1989, Figure 7.39].

Augmenting Paths

-  An **augmenting path** w.r.t. a given flow f (of a network G) is a directed path from s to t consisting of edges from G , but not necessarily in the same direction; each of these edges (v, u) satisfies exactly one of:
1. (v, u) is in the same direction as it is in G , and $f(v, u) < c(v, u)$. (*forward edge*)
 2. (v, u) is in the opposite direction in G (namely, $(u, v) \in E$), and $f(u, v) > 0$. (*backward edge*)
-  If there exists an augmenting path w.r.t. a flow f (f admits an augmenting path), then f is not maximum.

Augmenting Paths (cont.)

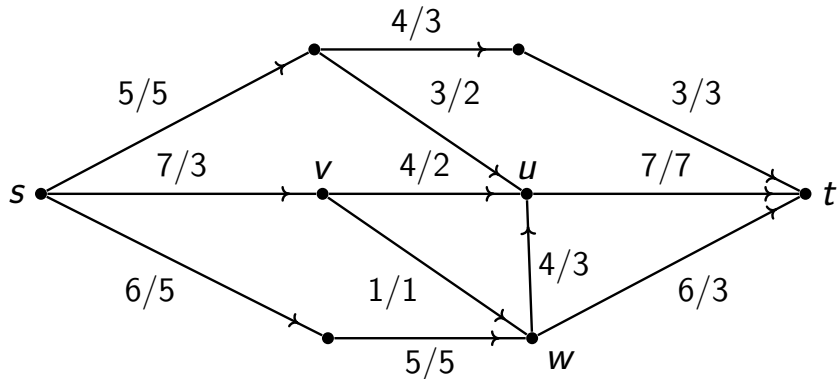


Figure: An example of a network with a (nonmaximum) flow.

Source: redrawn from [Manber 1989, Figure 7.40].

Augmenting Paths (cont.)

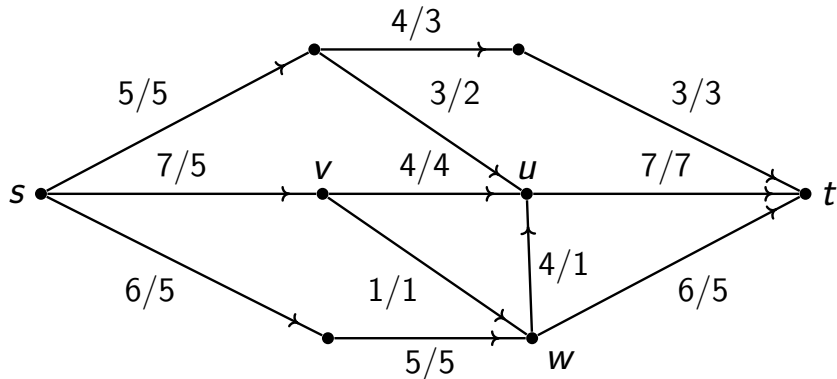


Figure: The result of augmenting the flow of Figure 7.40.

Source: redrawn from [Manber 1989, Figure 7.41].

Properties of Network Flows

Theorem (Augmenting-Path)

A flow f is maximum if and only if it admits no augmenting path.

A *cut* is a set of edges that separate s from t , or more precisely a set of the form $\{(v, w) \in E \mid v \in A \text{ and } w \in B\}$, where $B = V - A$ such that $s \in A$ and $t \in B$.

Theorem (Max-Flow Min-Cut)



The value of a maximum flow in a network is equal to the minimum capacity of a cut.

Properties of Network Flows (cont.)

Theorem (Integral-Flow)

If the capacities of all edges in the network are integers, then there is a maximum flow whose value is an integer.

Residual Graphs

-  The **residual graph** with respect to a network $G = (V, E)$ and a flow f is the network $R = (V, F)$, where F consists of all forward and backward edges and their capacities are given as follows:
1. $c_R(v, w) = c(v, w) - f(v, w)$ if (v, w) is a forward edge and
 2. $c_R(v, w) = f(w, v)$ if (v, w) is a backward edge.
-  An augmenting path is thus a regular directed path from s to t in the residual graph.

Residual Graphs (cont.)

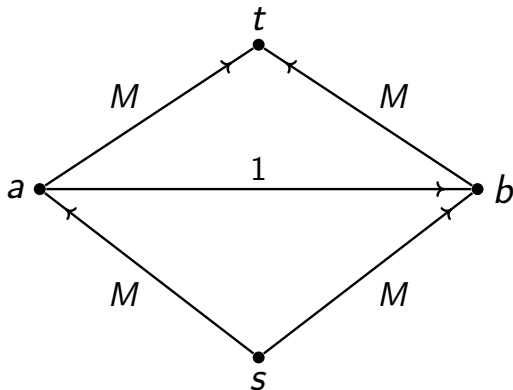


Figure: A bad example of network flow.

Source: redrawn from [Manber 1989, Figure 7.42].