

Algorithms 2022: Reduction

(Based on [Manber 1989])

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1 Introduction

Introduction

- The basic idea of *reduction* is to solve a problem with the solution to another “similar” problem.
- When Problem A can be reduced (efficiently) to Problem B , there are two consequences:
 - A solution to Problem B may be used to solve Problem A .
 - If A is known to be “hard”, then B is also necessarily “hard”.

/* A reduction should be reasonably efficient (this will be made precise in the topic of NP-completeness). Otherwise, one might be able to reduce a hard problem to a simpler one, by solving the more time-consuming part during the process of reduction and leaving the easier part to the second problem. */

- One should avoid the pitfall of reducing a problem to another that is too general or too hard.

2 Bipartite Matching

Matching

- Given an undirected graph $G = (V, E)$, a **matching** is a set of edges that do not share a common vertex.
- A **maximum** matching is one with the maximum number of edges.
- A **maximal** matching is one that cannot be extended by adding any other edge.

Bipartite Matching

- A bipartite graph $G = (V, E, U)$ is a graph with $V \cup U$ as the set of vertices and E as the set of edges such that
 - V and U are disjoint and
 - The edges in E connect vertices from V to vertices in U .

Problem 1. *Given a bipartite graph $G = (V, E, U)$, find a maximum matching in G .*

Bipartite Matching (cont.)

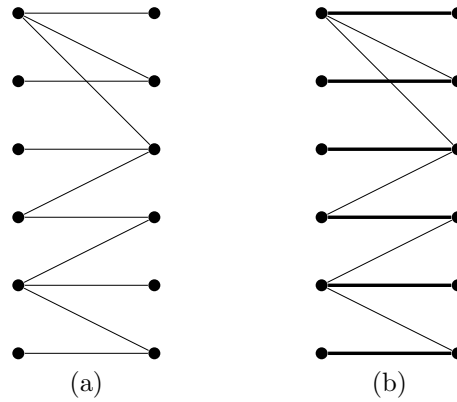


Figure: A bipartite graph and a maximum matching.

Source: adapted from [Manber 1989, Figure 7.37].

Bipartite Matching (cont.)

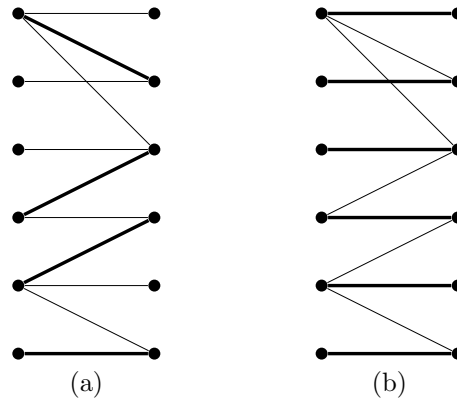


Figure: A maximal matching and a maximum matching.

Source: adapted from [Manber 1989, Figure 7.37].

3 Network Flows

Networks

- Consider a directed graph, or network, $G = (V, E)$ with two distinguished vertices: s (the source) with indegree 0 and t (the sink) with outdegree 0.
- Each edge e in E has an associated positive weight $c(e)$, called the *capacity* of e .

The Network Flow Problem

- A **flow** is a function f on E that satisfies the following two conditions:
 1. $0 \leq f(e) \leq c(e)$.
 2. $\sum_u f(u, v) = \sum_w f(v, w)$, for all $v \in V - \{s, t\}$.
- The **network flow problem** is to maximize the flow f for a given network G .

4 Bipartite Matching to Network Flow

Bipartite Matching to Network Flow

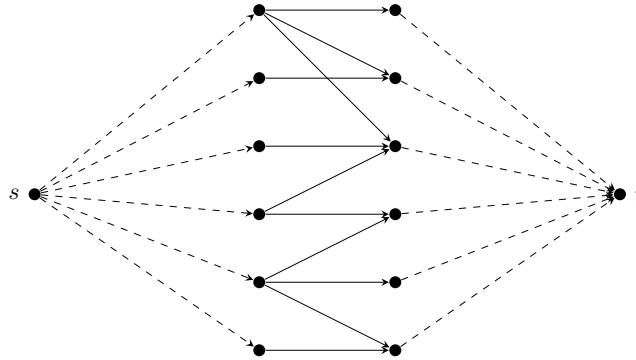


Figure: Reducing bipartite matching to network flow. Every edge has capacity 1.

Source: redrawn from [Manber 1989, Figure 7.39].

Bipartite Matching to Network Flow (cont.)

- Mapping from the input $G = (V, E, U)$ of the bipartite matching problem to the input $G' = (V', E')$ and c of the network flow problem:
 - The network is $G' = (V', E')$ where
 - * $V' = \{s\} \cup V \cup U \cup \{t\}$
 - * $E' = \{(s, v) \mid v \in V\} \cup E \cup \{(u, t) \mid u \in U\}$
 - The capacity for every $e \in E'$ is 1, i.e., $\forall e \in E', c(e) = 1$.
- Correspondence between the two solutions
 - A maximum flow f in G' defines a maximum matching M_f in G .
 - A maximum matching M in G induces a maximum flow f_M in G' .

5 Linear Programming

Notations

- Let \bar{v} denote a vector (v_1, v_2, \dots, v_n) of n constants or n variables.
- In the following, \bar{a} , \bar{b} , \bar{c} , and \bar{e} are vectors of n constants.
- And, \bar{x} and \bar{y} are vectors of n variables.
- The (inner or dot) product $\bar{a} \cdot \bar{x}$ of two vectors \bar{a} and \bar{x} is defined as follows:

$$\bar{a} \cdot \bar{x} = \sum_{i=1}^n a_i \cdot x_i$$

Linear Programming

- Objective function:

$$\bar{c} \cdot \bar{x}$$

- Equality constraints:

$$\begin{aligned}\bar{e}_1 \cdot \bar{x} &= d_1 \\ \bar{e}_2 \cdot \bar{x} &= d_2 \\ &\vdots \\ \bar{e}_m \cdot \bar{x} &= d_m\end{aligned}$$

- Inequality constraints may be turned into equality constraints by introducing *slack* variables.
- Non-negative constraints: $x_j \geq 0$, for all j in P , where P is a subset of $\{1, 2, \dots, n\}$.
- The goal is to *maximize* (or *minimize*) the value of the objective function, subject to the equality constraints.

6 Network Flow to Linear Programming

Network Flow to Linear Programming

- From the input $G = (V, E)$ and c of the network flow problem to the objective function and constraints of linear programming:

- Let x_1, x_2, \dots, x_n represent the flow values of the n edges.
- Objective function:

$$\sum_{i \in S} x_i$$

where S is the set of edges leaving the source.

- Inequality constraints:

$$x_i \leq c_i, \text{ for all } i, 1 \leq i \leq n$$

where c_i is the capacity of edge i .

- Equality constraints:

$$\sum_{i \text{ leaves } v} x_i - \sum_{j \text{ enters } v} x_j = 0, \text{ for every } v \in V \setminus \{s, t\}$$

- Non-negative constraints: $x_i \geq 0$, for all $i, 1 \leq i \leq n$.

/* If f is a maximum flow for $G = (V, E)$ and c , then $x_i = f(i)$, for $1 \leq i \leq n$, is a solution to the resulting linear programming problem.

Conversely, if $x_i = v_i$, for $1 \leq i \leq n$, is a solution to the resulting linear programming problem, then f with $f(i) = v_i$, for $1 \leq i \leq n$, is a maximum flow for $G = (V, E)$ and c . */