

Searching and Sorting

(Based on [Manber 1989])

Yih-Kuen Tsay

Department of Information Management
National Taiwan University

Searching a Sorted Sequence

Problem

Let x_1, x_2, \dots, x_n be a sequence of real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$. Given a real number z , we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.

Searching a Sorted Sequence

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Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

```
function Find (z, Left, Right) : integer;  
begin  
  if Left = Right then  
    if  $X[\textit{Left}] = z$  then Find := Left  
    else Find := 0  
  else  
    Middle :=  $\lceil \frac{\textit{Left} + \textit{Right}}{2} \rceil$ ;  
    if  $z < X[\textit{Middle}]$  then  
      Find := Find(z, Left, Middle - 1)  
    else  
      Find := Find(z, Middle, Right)  
  end
```

```
Algorithm Binary_Search (X, n, z);  
begin  
  Position := Find(z, 1, n);  
end
```

Binary Search: Alternative

```
function Find (z, Left, Right) : integer;  
begin  
  if Left > Right then  
    Find := 0  
  else  
    Middle :=  $\lceil \frac{\textit{Left} + \textit{Right}}{2} \rceil$ ;  
    if z = X[Middle] then  
      Find := Middle  
    else if z < X[Middle] then  
      Find := Find(z, Left, Middle - 1)  
    else  
      Find := Find(z, Middle + 1, Right)  
  end
```


How do the two algorithms compare?

Searching a Cyclically Sorted Sequence

Problem


Given a *cyclically sorted* list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).

Example 1:

 1 2 3 4 5 6 7 8
 [5 6 7 0 1 2 3 4]

 The 4th is the minimal element.

Example 2:

 1 2 3 4 5 6 7 8
 [0 1 2 3 4 5 6 7]

 The 1st is the minimal element.

Searching a Cyclically Sorted Sequence

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Given a *cyclically sorted* list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).

🌐 Example 1:

☀️ 1 2 3 4 5 6 7 8
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☀️ The 4th is the minimal element.

🌐 Example 2:

☀️ 1 2 3 4 5 6 7 8
 [0 1 2 3 4 5 6 7]

☀️ The 1st is the minimal element.

🌐 To cut the search space in half, what question should we ask?

Cyclic Binary Search

Algorithm Cyclic_Binary_Search (X, n);

begin

Position := Cyclic_Find(1, n);

end

function Cyclic_Find ($Left, Right$) : *integer*;

begin

if $Left = Right$ **then** *Cyclic_Find := Left*

else

Middle := $\lfloor \frac{Left+Right}{2} \rfloor$;

if $X[Middle] < X[Right]$ **then**

Cyclic_Find := Cyclic_Find(Left, Middle)

else

Cyclic_Find := Cyclic_Find(Middle + 1, Right)


end


“Fixpoints”


Problem


Given a sorted sequence of *distinct* integers a_1, a_2, \dots, a_n , determine whether there exists an index i such that $a_i = i$.


 Example 1:


$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$

 $a_4 = 4$ (there are more ...).

 Example 2:


$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$


 There is no i such that $a_i = i$.


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$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$

 $a_4 = 4$ (there are more ...).

 Example 2:


$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$

 There is no i such that $a_i = i$.

 Again, can we cut the search space in half by asking only one question?

A Special Binary Search

```
function Special_Find (Left, Right) : integer;  
begin  
  if Left = Right then  
    if A[Left] = Left then Special_Find := Left  
    else Special_Find := 0  
  else  
    Middle :=  $\lfloor \frac{Left+Right}{2} \rfloor$ ;  
    if A[Middle] < Middle then  
      Special_Find := Special_Find(Middle + 1, Right)  
    else  
      Special_Find := Special_Find(Left, Middle)  
end
```

A Special Binary Search (cont.)

```
Algorithm Special_Binary_Search ( $A, n$ );  
begin  
     $Position := Special\_Find(1, n)$ ;  
end
```

Stuttering Subsequence

Problem

Given two sequences $A (= a_1a_2 \cdots a_n)$ and $B (= b_1b_2 \cdots b_m)$, find the maximal value of i such that B^i is a subsequence of A .

- 🌐 If $B = xyzzx$, then $B^2 = xxyzzzzzx$, $B^3 = xxxyyzzzzzzzxxx$, etc.
- 🌐 B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- 🌐 For example, $B^2 = xxyzzzzzx$ is a subsequence of $xxzzyyyyxxzzzzzxxx$.

Stuttering Subsequence

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- 🌐 If B^j is a subsequence of A , then B^i is a subsequence of A , for $1 \leq i \leq j$.

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- 🌐 For example, $B^2 = xxyzzzzzx$ is a subsequence of $xxzzyyyyxxzzzzzxxx$.
- 🌐 If B^j is a subsequence of A , then B^i is a subsequence of A , for $1 \leq i \leq j$.
- 🌐 The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

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- 🌐 Sequential search: try 1, 2, 3, etc. sequentially.

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Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

Think of the base case in a reversed induction.

Interpolation Search

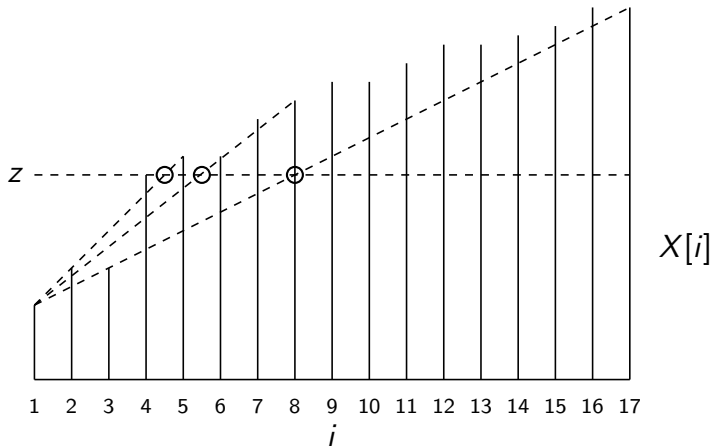
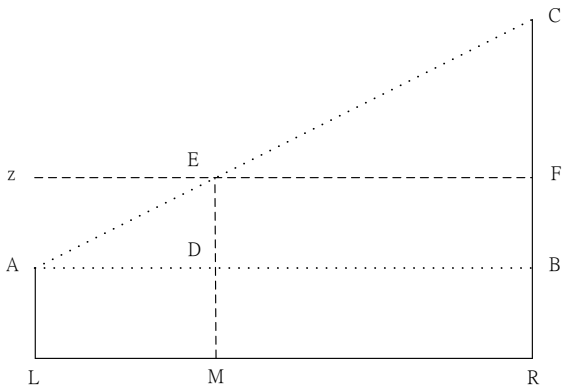


Figure: Interpolation search.

Source: redrawn from [Manber 1989, Figure 6.4].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{ so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

```
function Int_Find (z, Left, Right) : integer;  
begin  
  if  $X[Left] = z$  then Int_Find := Left  
  else if Left = Right or  $X[Left] = X[Right]$  then  
    Int_Find := 0  
  else  
     $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$ ;  
    if  $z < X[Next\_Guess]$  then  
      Int_Find := Int_Find(z, Left, Next_Guess - 1)  
    else  
      Int_Find := Int_Find(z, Next_Guess, Right)  
  end
```

Interpolation Search (cont.)


```
Algorithm Interpolation_Search ( $X, n, z$ );  
begin  
    if  $z < X[1]$  or  $z > X[n]$  then  $Position := 0$   
    else  $Position := Int\_Find(z, 1, n)$ ;  
end
```


Problem

Given n numbers x_1, x_2, \dots, x_n , arrange them in increasing order. In other words, find a sequence of distinct indices $1 \leq i_1, i_2, \dots, i_n \leq n$, such that $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$.

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

Using Balanced Search Trees

-  Balanced search trees, such as AVL trees, may be used for sorting:
1. Create an empty tree.
 2. Insert the numbers one by one to the tree.
 3. Traverse the tree and output the numbers.

Using Balanced Search Trees

- 🌐 Balanced search trees, such as AVL trees, may be used for sorting:
 1. Create an empty tree.
 2. Insert the numbers one by one to the tree.
 3. Traverse the tree and output the numbers.
- 🌐 What's the time complexity? Suppose we use an AVL tree.

Radix Sort

Algorithm Straight_Radix (X, n, k);

begin

put all elements of X in a queue GQ ;

for $i := 1$ **to** d **do**

initialize queue $Q[i]$ to be empty

for $i := k$ **downto** 1 **do**

while GQ is not empty **do**

pop x from GQ ;

$d :=$ the i -th digit of x ;

insert x into $Q[d]$;

for $t := 1$ **to** d **do**

insert $Q[t]$ into GQ ;

for $i := 1$ **to** n **do**

pop $X[i]$ from GQ

end

Radix Sort

```
Algorithm Straight_Radix ( $X, n, k$ );  
begin  
    put all elements of  $X$  in a queue  $GQ$ ;  
    for  $i := 1$  to  $d$  do  
        initialize queue  $Q[i]$  to be empty  
    for  $i := k$  downto  $1$  do  
        while  $GQ$  is not empty do  
            pop  $x$  from  $GQ$ ;  
             $d :=$  the  $i$ -th digit of  $x$ ;  
            insert  $x$  into  $Q[d]$ ;  
        for  $t := 1$  to  $d$  do  
            insert  $Q[t]$  into  $GQ$ ;  
    for  $i := 1$  to  $n$  do  
        pop  $X[i]$  from  $GQ$   
end
```

Time complexity: $O(nk)$.

Merge Sort

Algorithm Mergesort (X, n);
begin $M_Sort(1, n)$ **end**

procedure M_Sort ($Left, Right$);
begin

if $Right - Left = 1$ **then**

if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$

else if $Left \neq Right$ **then**

$Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;

$M_Sort(Left, Middle - 1)$;

$M_Sort(Middle, Right)$;

Merge Sort (cont.)

```
i := Left; j := Middle; k := 0;
while (i ≤ Middle − 1) and (j ≤ Right) do
    k := k + 1;
    if  $X[i] \leq X[j]$  then
         $TEMP[k] := X[i]$ ; i := i + 1
    else  $TEMP[k] := X[j]$ ; j := j + 1;
if j > Right then
    for t := 0 to Middle − 1 − i do
         $X[Right - t] := X[Middle - 1 - t]$ 
for t := 0 to k − 1 do
     $X[Left + t] := TEMP[1 + t]$ 
end
```

Merge Sort (cont.)

```
i := Left; j := Middle; k := 0;
while (i ≤ Middle − 1) and (j ≤ Right) do
    k := k + 1;
    if X[i] ≤ X[j] then
        TEMP[k] := X[i]; i := i + 1
    else TEMP[k] := X[j]; j := j + 1;
if j > Right then
    for t := 0 to Middle − 1 − i do
        X[Right − t] := X[Middle − 1 − t]
    for t := 0 to k − 1 do
        X[Left + t] := TEMP[1 + t]
end
```

Time complexity: $O(n \log n)$.

Merge Sort (cont.)

| | | | | | | | | | | | | | | | |
|---|---|---|---|----|----|----|----|----|----|----|----|---|----|----|----|
| 6 | 2 | 8 | 5 | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| ② | ⑥ | 8 | 5 | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 2 | 6 | ⑤ | ⑧ | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| ② | ⑤ | ⑥ | ⑧ | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 2 | 5 | 6 | 8 | ⑨ | ⑩ | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 2 | 5 | 6 | 8 | 9 | 10 | ① | ⑫ | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 2 | 5 | 6 | 8 | ① | ⑨ | ⑩ | ⑫ | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| ① | ② | ⑤ | ⑥ | ⑧ | ⑨ | ⑩ | ⑫ | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | ⑦ | ⑮ | 3 | 13 | 4 | 11 | 16 | 14 |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | 7 | 15 | ③ | ⑬ | 4 | 11 | 16 | 14 |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | ③ | ⑦ | ⑬ | ⑮ | 4 | 11 | 16 | 14 |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | 3 | 7 | 13 | 15 | ④ | ⑪ | 16 | 14 |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | 3 | 7 | 13 | 15 | 4 | 11 | ⑭ | ⑰ |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | 3 | 7 | 13 | 15 | ④ | ⑪ | ⑭ | ⑰ |
| 1 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | ③ | ④ | ⑦ | ⑪ | ⑬ | ⑭ | ⑮ | ⑰ |
| ① | ② | ③ | ④ | ⑤ | ⑥ | ⑦ | ⑧ | ⑨ | ⑩ | ⑪ | ⑫ | ⑬ | ⑭ | ⑮ | ⑰ |

Figure: An example of mergesort.

Source: redrawn from [Manber 1989, Figure 6.8].

Quick Sort

```
Algorithm Quicksort ( $X, n$ );  
begin  
     $Q\_Sort(1, n)$   
end  
  
procedure  $Q\_Sort$  ( $Left, Right$ );  
begin  
    if  $Left < Right$  then  
         $Partition(X, Left, Right)$ ;  
         $Q\_Sort(Left, Middle - 1)$ ;  
         $Q\_Sort(Middle + 1, Right)$   
    end  
end
```

Quick Sort

```
Algorithm Quicksort ( $X, n$ );  
begin  
     $Q\_Sort(1, n)$   
end  
  
procedure  $Q\_Sort$  ( $Left, Right$ );  
begin  
    if  $Left < Right$  then  
         $Partition(X, Left, Right)$ ;  
         $Q\_Sort(Left, Middle - 1)$ ;  
         $Q\_Sort(Middle + 1, Right)$   
    end
```

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

Quick Sort (cont.)

Algorithm Partition($X, Left, Right$);

begin

$pivot := X[Left]$;

$L := Left$; $R := Right$;

while $L < R$ **do**

while $X[L] \leq pivot$ and $L \leq Right$ **do** $L := L + 1$;

while $X[R] > pivot$ and $R \geq Left$ **do** $R := R - 1$;

if $L < R$ **then** $swap(X[L], X[R])$;

$Middle := R$;

$swap(X[Left], X[Middle])$

end

Quick Sort (cont.)

| | | | | | | | | | | | | | | | |
|---|---|---|---|----|---|----|---|----|---|----|----|---|----|----|----|
| 6 | 2 | 8 | 5 | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 6 | 2 | ④ | 5 | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | ⑧ | 11 | 16 | 14 |
| 6 | 2 | 4 | 5 | ③ | 9 | 12 | 1 | 15 | 7 | ⑩ | 13 | 8 | 11 | 16 | 14 |
| 6 | 2 | 4 | 5 | 3 | ① | 12 | ⑨ | 15 | 7 | 10 | 13 | 8 | 11 | 16 | 14 |
| ① | 2 | 4 | 5 | 3 | ⑥ | 12 | 9 | 15 | 7 | 10 | 13 | 8 | 11 | 16 | 14 |

Figure: Partition of an array around the pivot 6.

Source: redrawn from [Manber 1989, Figure 6.10].

Quick Sort (cont.)

| | | | | | | | | | | | | | | | |
|---|---|---|---|----|---|----|---|----|---|----|----|----|----|----|----|
| 6 | 2 | 8 | 5 | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 1 | 2 | 4 | 5 | 3 | ⑥ | 12 | 9 | 15 | 7 | 10 | 13 | 8 | 11 | 16 | 14 |
| ① | 2 | 4 | 5 | 3 | ⑥ | 12 | 9 | 15 | 7 | 10 | 13 | 8 | 11 | 16 | 14 |
| ① | ② | 4 | 5 | 3 | ⑥ | 12 | 9 | 15 | 7 | 10 | 13 | 8 | 11 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 12 | 9 | 15 | 7 | 10 | 13 | 8 | 11 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 8 | 9 | 11 | 7 | 10 | ⑫ | 13 | 15 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 7 | ⑧ | 11 | 9 | 10 | ⑫ | 13 | 15 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 7 | ⑧ | 10 | 9 | ⑪ | ⑫ | 13 | 15 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 7 | ⑧ | 9 | ⑩ | ⑪ | ⑫ | 13 | 15 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 7 | ⑧ | 9 | ⑩ | ⑪ | ⑫ | ⑬ | 15 | 16 | 14 |
| ① | ② | 3 | ④ | 5 | ⑥ | 7 | ⑧ | 9 | ⑩ | ⑪ | ⑫ | ⑬ | 14 | ⑮ | 16 |

Figure: An example of quicksort.

Source: redrawn from [Manber 1989, Figure 6.12].

Average-Case Complexity of Quick Sort

🌐 When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i - 1) + T(n - i), \text{ where } n \geq 2.$$

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i - 1) + T(n - i), \text{ where } n \geq 2.$$

The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \frac{1}{n} \sum_{i=1}^n (T(i - 1) + T(n - i)) \\ &= n - 1 + \frac{1}{n} \sum_{i=1}^n T(i - 1) + \frac{1}{n} \sum_{i=1}^n T(n - i) \\ &= n - 1 + \frac{1}{n} \sum_{j=0}^{n-1} T(j) + \frac{1}{n} \sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \frac{2}{n} \sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history,
 $T(n) = O(n \log n)$.

Heap Sort

```
Algorithm Heapsort ( $A, n$ );  
begin  
    Build_Heap( $A$ );  
    for  $i := n$  downto 2 do  
        swap( $A[1], A[i]$ );  
        Rearrange_Heap( $i - 1$ )  
end
```

Heap Sort

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begin  
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    for  $i := n$  downto 2 do  
        swap( $A[1], A[i]$ );  
        Rearrange_Heap( $i - 1$ )  
end
```

Time complexity: $O(n \log n)$

Heap Sort (cont.)

```
procedure Rearrange_Heap (k);  
begin  
    parent := 1;  
    child := 2;  
    while child ≤ k - 1 do  
        if  $A[\textit{child}] < A[\textit{child} + 1]$  then  
            child := child + 1;  
        if  $A[\textit{child}] > A[\textit{parent}]$  then  
            swap( $A[\textit{parent}]$ ,  $A[\textit{child}]$ );  
            parent := child;  
            child := 2 * child  
        else child := k  
end
```

Heap Sort (cont.)

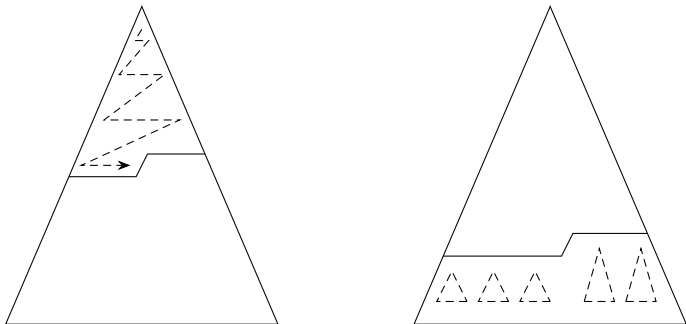


Figure: Top down and bottom up heap construction.

Source: redrawn from [Manber 1989, Figure 6.14].

Heap Sort (cont.)

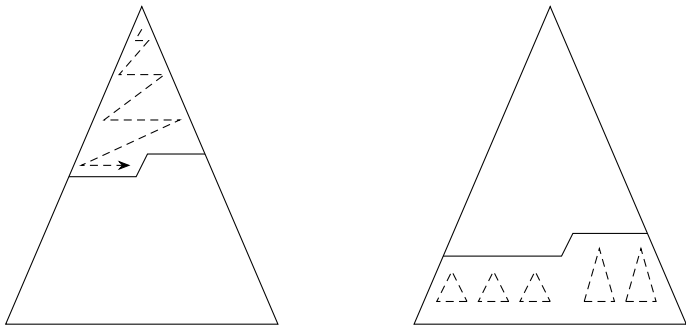


Figure: Top down and bottom up heap construction.

Source: redrawn from [Manber 1989, Figure 6.14].

How do the two approaches compare?

Building a Heap Bottom Up

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 6 | 2 | 8 | 5 | 10 | 9 | 12 | 1 | 15 | 7 | 3 | 13 | 4 | 11 | 16 | 14 |
| 6 | 2 | 8 | 5 | 10 | 9 | 12 | ⑭ | 15 | 7 | 3 | 13 | 4 | 11 | 16 | ① |
| 6 | 2 | 8 | 5 | 10 | 9 | ⑯ | 14 | 15 | 7 | 3 | 13 | 4 | 11 | ⑫ | 1 |
| 6 | 2 | 8 | 5 | 10 | ⑬ | 16 | 14 | 15 | 7 | 3 | ⑨ | 4 | 11 | 12 | 1 |
| 6 | 2 | 8 | 5 | 10 | 13 | 16 | 14 | 15 | 7 | 3 | 9 | 4 | 11 | 12 | 1 |
| 6 | 2 | 8 | ⑮ | 10 | 13 | 16 | 14 | ⑤ | 7 | 3 | 9 | 4 | 11 | 12 | 1 |
| 6 | 2 | ⑯ | 15 | 10 | 13 | ⑫ | 14 | 5 | 7 | 3 | 9 | 4 | 11 | ⑧ | 1 |
| 6 | ⑮ | 16 | ⑭ | 10 | 13 | 12 | ② | 5 | 7 | 3 | 9 | 4 | 11 | 8 | 1 |
| ⑯ | 15 | ⑬ | 14 | 10 | ⑨ | 12 | 2 | 5 | 7 | 3 | ⑥ | 4 | 11 | 8 | 1 |

Figure: An example of building a heap bottom up.

Source: adapted from [Manber 1989, Figure 6.15].

A Lower Bound for Sorting

- 🌐 A **lower bound** for a particular problem is a proof that *no algorithm* can solve the problem better.
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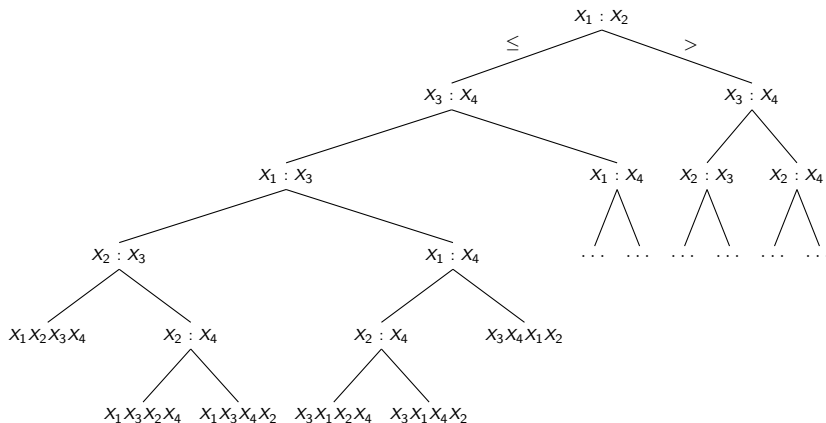
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Proof idea: there must be at least $n!$ leaves in the decision tree, one for each possible outcome.

Is the lower bound contradictory to the time complexity of radix sort?

A Lower Bound for Sorting (cont.)

A decision tree (partly shown) for the merge sort with $X_1X_2X_3X_4$ as input:



Note: in total, there should be $4! = 24$ leaves, only six of which are shown.

Order Statistics: Minimum and Maximum



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- 🌐 The obvious solution requires $(n - 1) + (n - 2) (= 2n - 3)$ comparisons between elements.
- 🌐 Can we do better? (Which comparisons could have been avoided?)

Order Statistics: K th-Smallest

Problem

Given a sequence $S = x_1, x_2, \dots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the k th-smallest element in S .

Order Statistics: K th-Smallest (cont.)

```
procedure Select (Left, Right, k);  
begin  
  if Left = Right then  
    Select := Left  
  else Partition(X, Left, Right);  
    let Middle be the output of Partition;  
    if Middle - Left + 1  $\geq$  k then  
      Select(Left, Middle, k)  
    else  
      Select(Middle + 1, Right, k - (Middle - Left + 1))  
end
```

```
Algorithm Selection (X, n, k);  
begin  
  if (k < 1) or (k > n) then print "error"  
  else S := Select(1, n, k)  
end
```


Order Statistics: K th-Smallest (cont.)

The nested “if” statement may be simplified:

```
procedure Select (Left, Right, k);  
begin  
  if Left = Right then  
    Select := Left  
  else Partition(X, Left, Right);  
    let Middle be the output of Partition;  
    if Middle  $\geq$  k then  
      Select(Left, Middle, k)  
    else  
      Select(Middle + 1, Right, k)  
end
```

Finding a Majority

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Given a sequence of numbers, find the majority in the sequence or determine that none exists.

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Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

What if they are equal?

Finding a Majority (cont.)

```
Algorithm Majority ( $X, n$ );  
begin  
   $C := X[1]; M := 1;$   
  for  $i := 2$  to  $n$  do  
    if  $M = 0$  then  
       $C := X[i]; M := 1$   
    else  
      if  $C = X[i]$  then  $M := M + 1$   
      else  $M := M - 1;$ 
```

Finding a Majority (cont.)

```
if  $M = 0$  then Majority := -1  
else  
    Count := 0;  
    for  $i := 1$  to  $n$  do  
        if  $X[i] = C$  then Count := Count + 1;  
    if  $Count > n/2$  then Majority :=  $C$   
    else Majority := -1  
end
```

Finding a Majority (cont.)

```
if  $M = 0$  then  $Majority := -1$   
else  
     $Count := 0;$   
    for  $i := 1$  to  $n$  do  
        if  $X[i] = C$  then  $Count := Count + 1;$   
    if  $Count > n/2$  then  $Majority := C$   
    else  $Majority := -1$   
end
```

Time complexity: $O(n)$.