Homework Assignment #1

Due Time/Date

2:20PM Tuesday, September 13, 2022. Late submission will be penalized by 20% for each working day overdue.

How to Submit

Please use a word processor or scan hand-written answers to produce a single PDF file. Name your file according to this pattern: "b107050xx-hw1". Upload the PDF file to the NTU COOL site for Algorithms 2022. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

There are five problems in this assignment, each accounting for 20 points. You must use *induction* for all proofs. (Note: problems marked with "(X.XX)" are taken from [Manber 1989] with probable adaptation.)

1. (2.10) Find an expression for the sum of the i-th row of the following triangle, which is called the **Pascal triangle**, and prove the correctness of your claim. The sides of the triangle are 1s, and each other entry is the sum of the two entries immediately above it.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

2. The Harmonic series $H(k)$ is defined by $H(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k}$. Prove that $H(2^n) \geq 1 + \frac{n}{2}$, for all $n \geq 0$ (which implies that $H(k)$ diverges).

3. (2.14) Consider the following series: 1, 2, 3, 4, 5, 10, 20, 40, ..., which starts as an arithmetic series, but after the first 5 terms becomes a geometric series. Prove that any positive integer can be written as a sum of distinct numbers from this series.

4. (2.37) Consider the recurrence relation for Fibonacci numbers $F(n) = F(n-1)+F(n-2)$. Without solving this recurrence, compare $F(n)$ to $G(n)$ defined by the recurrence $G(n) = G(n-1)+G(n-2)+1$. It seems obvious that $G(n) > F(n)$ (because of the extra 1). Yet the following is a seemingly valid proof (by induction) that $G(n) = F(n) - 1$. We assume, by induction, that $G(k) = F(k)-1$ for all k such that $1 \leq k \leq n$, and we consider $G(n+1)$:

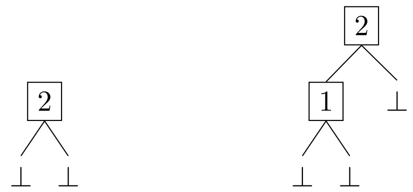
$$G(n+1) = G(n) + G(n-1) + 1 = F(n) - 1 + F(n-1) - 1 + 1 = F(n+1) - 1$$

   What is wrong with this proof?

5. The set of all binary trees that store non-negative integer key values may be defined inductively as follows.

- The empty tree, denoted $\perp$, is a binary tree, storing no key value.
- If t_l and t_r are binary trees, then $node(k, t_l, t_r)$, where $k \in Z$ and $k \geq 0$, is a also binary tree with the root storing key value k.

So, for instance, $node(2, \perp, \perp)$ is a single-node binary tree storing key value 2 and $node(2, node(1, \perp, \perp), \perp)$ is a binary tree with two nodes — the root and its left child, storing key values 2 and 1, repsectively. Pictorially, they may be depicted as below.



$node(2, \perp, \perp)$ $\quad$ $node(2, node(1, \perp, \perp), \perp)$

(a) (10 points) Define inductively a function Max that determines the largest of all key values of a binary tree. Let $Max(\perp) = 0$, though the empty tree does not store any key value. (Note: use the usual mathematical notations; do not write a computer program.)

(b) (10 points) Suppose, to differentiate the empty tree from a non-empty tree whose largest key value happens to be 0, we require that $Max(\perp) = -1$. Give another definition for Max that meets this requirement; again, induction should be used somewhere in the definition.