Bounded Model Checking

(Based on [Biere *et al.* 1999, Benedetti and Cimatti 2003])

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Outline

- Introduction
- An Illustrative Example
- Part I: Bounded Model Checking for LTL (with future only)
- Part II: Bounded Model Checking for LTL with Past (or Full PTL)
- References:
  [Biere *et al.*] A. Biere, A. Cimatti, E. Clarke, and Y. Zhu, “Symbolic Model Checking without BDD,” TACAS 1999, LNCS1579.
  [Benedetti and Cimatti] M. Benedetti and A. Cimatti, “Bounded Model Checking for Past LTL,” TACAS 2003, LNCS 2619.

Introduction

- In *symbolic model checking*, BDDs had traditionally been used for boolean encodings.
- Drawbacks of BDDs:
  - For large systems (with over a few hundred boolean variables), they can be prohibitively large.
  - Selecting the right variable ordering is often time-consuming or needs manual intervention.
- Propositional decision procedures, or SAT solvers, also operate on boolean expressions, but do not use canonical forms.
- SAT solvers can handle thousands of variables or even more.

Introduction (cont.)

- Basic ideas of bounded model checking (BMC):
  - Consider counterexamples of a particular length k.
  - Generate a propositional formula that is satisfiable iff such a counterexample exists.
  - The propositional formula can be tested for satisfiability by a SAT solver.
- Advantages of BMC:
  - It finds counterexamples very fast.
  - It finds counterexamples of minimal length.
  - It uses much less space than BDD-based approaches.
  - It does not need a manually selected variable ordering or time-consuming dynamic reordering.

An Example

- Consider a three-bit shift register.
- Let $M = \langle X, I, T \rangle$ be its state machine:
  - $X \triangleq \{x[0], x[1], x[2]\}$ contains the three bits.
  - $I(X) \triangleq true$, posing no restriction on the initial states.
  - $T(X, X') \triangleq (x'[0] \Leftrightarrow x[1]) \wedge (x'[1] \Leftrightarrow x[2]) \wedge x'[2]$.
- Suppose we want to check if eventually all three bits are set to 0, i.e., if LTL formula $p \triangleq \diamond(\neg x[0] \wedge \neg x[1] \wedge \neg x[2])$ holds on all paths in M.
- To do so, we search for a path in M such that $\neg p \triangleq \square(x[0] \vee x[1] \vee x[2])$ on the path.
- If we succeed, then p does not hold on all paths; otherwise, it does.

An Example (cont.)

- We look for (looping) paths with at most $k+1$ states, for instance $k=2$.
- Let X_i denote the set $\{x_i[0], x_i[1], x_i[2]\}$.
- The first 3 states of such a path can be characterized by the following boolean formula:

$$f_M \triangleq I(X_0) \wedge T(X_0, X_1) \wedge T(X_1, X_2)$$

- A witness for $\neg p$ must contain a loop from X_2 back to X_0, X_1, or X_2:

$$L_i \triangleq T(X_2, X_i)$$

- The path must fulfill the constraints imposed by $\neg p$:

$$S_i \triangleq x_i[0] \vee x_i[1] \vee x_i[2]$$

An Example (cont.)

- The following formula is satisfiable iff there is a counterexample of length 2 for p.

$$f_M \wedge \bigvee_{i=0}^{2} L_i \wedge \bigwedge_{i=0}^{2} S_i$$

- Here is a satisfying assignment:

$$\begin{aligned} & x_0[0] = x_0[1] = x_0[2] \\ = \; & x_1[0] = x_1[1] = x_1[2] \\ = \; & x_2[0] = x_2[1] = x_2[2] \\ = \; & 1. \end{aligned}$$

Part I:

Bounded Model Checking for LTL

Kripke Structures

- A Kripke structure is a tuple $M = (S, I, T, L)$ with
  - a finite set of states S,
  - the set of initial states $I \subseteq S$,
  - a transition relation between states $T \subseteq S \times S$, and
  - the labeling of the states $L : S \to \mathscr{P}(A)$ with atomic propositions A.
- Every state of M is required to have a successor.
- We write $s \to t$ for $(s, t) \in T$.
- For an infinite sequence π of states $s_0, s_1, \ldots$, we define
  - $\pi(i) = s_i$
  - $\pi^i = s_i, s_{i+1}, \ldots$.
- An infinite sequence π is a path if $\pi(i) \to \pi(i+1)$ for all $i \in \mathbb{N}$.

Linear Temporal Logic (LTL)

- Let M be a Kripke structure, π be a path in M, and f be an LTL formula (in negation normal form).
- $\pi \models f$ (f is valid along π) is defined as follows:

$\pi \models p$	iff	$p \in L(\pi(0))$
$\pi \models \neg p$	iff	$p \notin L(\pi(0))$
$\pi \models f \wedge g$	iff	$\pi \models f$ and $\pi \models g$
$\pi \models f \vee g$	iff	$\pi \models f$ or $\pi \models g$
$\pi \models \Box f$	iff	$\forall j \in [0, \infty).\pi^j \models f$
$\pi \models \Diamond f$	iff	$\exists j \in [0, \infty).\pi^j \models f$
$\pi \models \bigcirc f$	iff	$\pi^1 \models f$
$\pi \models f \,\mathcal{U}\, g$	iff	$\exists j \in [0, \infty).(\pi^j \models g$ and $\forall k \in [0, j).\pi^k \models f)$
$\pi \models f \,\mathcal{R}\, g$	iff	$\forall j \in [0, \infty).(\pi^j \models g$ or $\exists k \in [0, j).\pi^k \models f)$

Model Checking

- An LTL formula f is valid in a Kripke structure M, denoted as $M \models$ **A** f, iff $\pi \models f$ for all paths π in M with $\pi(0) \in I$.
- An LTL formula f is satisfiable in a Kripke structure M, denoted as $M \models$ **E** f, iff there is a path π in M such that $\pi \models f$ and $\pi(0) \in I$.
- Given a Kripke structure M and an LTL formula f, the model checking problem is to determine whether $M \models$ **A** f, which is equivalent to determine whether $M \not\models$ **E** $\neg f$.
- In the following, the problem is restricted to find a witness for formulae of the form **E** f.

Bounded Model Checking

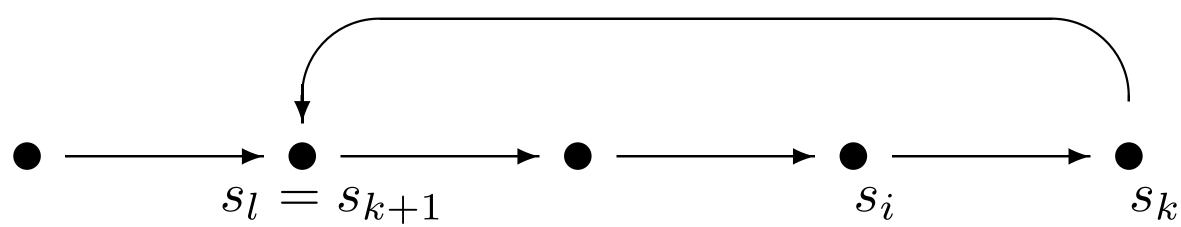
- Consider only a finite prefix of a path that may be a witness of **E** f.
- We restrict the length of the prefix to a certain bound k.
- Generate a propositional formula that is satisfiable iff there is a witness within the bound k.
- The propositional formula can be solved by a SAT solver.
- If there is no witness within bound k, we increase the bound and look for longer and longer possible witnesses.

Infinite Paths from the Prefix

- Though the prefix of a path is finite, it still might represent an infinite path if there is a back loop from the last state of the prefix to any of the previous states.
- If there is no such back loop, then the prefix does not say anything about the infinite behavior of the path.
- Only a prefix with a back loop can represent a witness for $\Box f$.

Loops

- A path π is a (k,l)-loop for $l \leq k$ if
  - $\pi(k) \rightarrow \pi(l)$ and
  - $\pi = u \cdot v^\omega$ with
    - $u = \pi(0), \ldots, \pi(l-1)$ and
    - $v = \pi(l), \ldots, \pi(k)$



- A path π is a k-loop if there is an $l \in \mathbb{N}$ with $l \leq k$ for which π is a (k,l)-loop.

Bounded Semantics

- In bounded semantics, we only consider a finite prefix of a path which may or may not be a loop.
- In particular, we only use the first $k + 1$ states of a path to determine the validity of a formula along that path.
- The bounded semantics $\pi \models_k f$ states that the LTL formula f is valid along the path π with bound k.

Bounded Semantics for a Loop

- Let $k \in \mathbb{N}$ and π be a k-loop.
- $\pi \models_k f$ iff $\pi \models f$.
- This is so, because all information about π is contained in the prefix of length k.

Bounded Semantics without a Loop

- Let $k \in \mathbb{N}$ and π be a path that is not a k-loop.
- $\pi \models_k f$ iff $(\pi, 0) \models_k f$ where

$(\pi, i) \models_k p$	iff	$p \in L(\pi(i))$
$(\pi, i) \models_k \neg p$	iff	$p \notin L(\pi(i))$
$(\pi, i) \models_k f \wedge g$	iff	$(\pi, i) \models_k f$ and $(\pi, i) \models_k g$
$(\pi, i) \models_k f \vee g$	iff	$(\pi, i) \models_k f$ or $(\pi, i) \models_k g$
$(\pi, i) \models_k \Box f$	iff	$false$
$(\pi, i) \models_k \Diamond f$	iff	$\exists j \in [i, k].(\pi, j) \models_k f$
$(\pi, i) \models_k \bigcirc f$	iff	$i < k$ and $(\pi, i+1) \models_k f$
$(\pi, i) \models_k f \,\mathcal{U}\, g$	iff	$\exists j \in [i, k].((\pi, j) \models_k g$ and $\forall n \in [i, j).(\pi, n) \models_k f)$
$(\pi, i) \models_k f \,\mathcal{R}\, g$	iff	$\exists j \in [i, k].((\pi, j) \models_k f$ and $\forall n \in [i, j].(\pi, n) \models_k g)$

- Note: $(\pi, i) \models_k f$ is written as $\pi \models_k^i f$ in the paper.

Bounded Semantics without a Loop (cont.)

- Note that the bounded semantics without a loop imply that the following two dualities no longer hold:
  - the duality of $\Box$ and $\Diamond$ ($\neg\Box f = \Diamond\neg f$), and
  - the duality of $\mathcal{U}$ and $\mathcal{R}$ ($\neg(f\ \mathcal{U}\ g) = (\neg f)\ \mathcal{R}\ (\neg g)$).

Reduce to Bounded Model Checking

- **Lemma 1** *Let* h *be an LTL formula and* π *a path, then* $\pi \models_k h \Rightarrow \pi \models h$.
- **Lemma 2** *Let* f *be an LTL formula and* M *a Kripke structure. If* $M \models$ ***E*** f *then there exists* $k \in \mathbb{N}$ *with* $M \models_k$ ***E*** f.
- **Theorem 3** *Let* f *be an LTL formula and* M *a Kripke structure. Then* $M \models$ ***E*** f *iff there exists* $k \in \mathbb{N}$ *with* $M \models_k$ ***E*** f.

Proof of Lemma 1

Let h be an LTL formula and π a path, then $\pi \models_k h \Rightarrow \pi \models h$.

- Case 1: π is a k-loop.
  - The conclusion follows by the definition.
- Case 2: π is not a loop.
  - Prove by induction over the structure of f and $i \leq k$ the stronger property $\pi \models_k^i h \Rightarrow \pi^i \models h$.

Proof of Lemma 1 (cont.)

$$
\begin{array}{ll}
 & \pi \models_k^i f \mathcal{R} g \\
\Leftrightarrow & \exists j \in [i,k].(\pi \models_k^j f \text{ and } \forall n \in [i,j].\pi \models_k^n g) \\
\Rightarrow & \exists j \in [i,k].(\pi^j \models f \text{ and } \forall n \in [i,j].\pi^n \models g) \\
\Rightarrow & \exists j \in [i,\infty].(\pi^j \models f \text{ and } \forall n \in [i,j].\pi^n \models g) \\
\Rightarrow & \exists j' \in [0,\infty).(\pi^{i+j'} \models f \text{ and } \forall n' \in [0,j'].\pi^{i+n'} \models g) \\
 & (\text{with } j' = j - i \text{ and } n' = n - i) \\
\Rightarrow & \exists j \in [0,\infty).[(\pi^i)^j \models f \text{ and } \forall n \in [0,j].(\pi^i)^n \models g] \\
\Rightarrow & \forall n \in [0,\infty).[(\pi^i)^n \models g \text{ or } \exists j \in [0,n).(\pi^i)^j \models f] \\
 & (\text{see next slide}) \\
\Rightarrow & \pi^i \models f \mathcal{R} g
\end{array}
$$

Proof of Lemma 1 (cont.)

$\exists m[\pi^m \models f$ and $\forall l, l \leq m.\pi^l \models g] \Rightarrow \forall n[\pi^n \models g$ or $\exists j, j < n.\pi^j \models f]$

- Assume that m is the smallest number such that $\pi^m \models f$ and $\pi^l \models g$ for all l with $l \leq m$.
- Case 1: $n > m$.
  - Based on the assumption, there exists $j < n$ such that $\pi^j \models f$ (choose $j = m$).
- Case 2: $n \leq m$.
  - Because $\pi^l \models g$ for all $l \leq m$ we have $\pi^n \models g$ for all $n \leq m$.

Proof of Lemma 2

Let f be an LTL formula and M a Kripke structure. If $M \models \mathbf{E}\ f$ then there exists $k \in \mathbb{N}$ with $M \models_k \mathbf{E}\ f$.

- If f is satisfiable in M, then there exists a path in the product structure of M and the tableau of f that starts with an initial state and ends with a cycle in the strongly connected component of fair states.
- This path can be chosen to be a k-loop with k bounded by $|S| \cdot 2^{|f|}$ which is the size of the product structure.
- If we project this path onto its first component, the original Kripke structure, then we get a path π that is a k-loop and in addition fulfills $\pi \models f$.
- By definition of the bounded semantics this also implies $\pi \models_k f$.

From BMC to SAT

- Given a Kripke structure M, an LTL formula f, and a bound k, we will construct a propositional formula $[\![M, f]\!]_k$.
- The bounded model checking problem can be reduced in polynomial time to propositional satisfiability.
  - The size of $[\![M, f]\!]_k$ is polynomial in the size of f if common sub-formulae are shared.
  - It is quadratic in k and linear in the size of the propositional formulae for T, I, and the $p \in A$.

Unfolding the Transition Relation

- For a Kripke structure M and $k \in \mathbb{N}$,

$$[\![M]\!]_k \triangleq I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1})$$

Trans. of an LTL formula without a Loop

- For an LTL formula f and $k, i \in \mathbb{N}$, with $i \leq k$,

$$
\begin{array}{lcl}
[\![p]\!]_k^i & \triangleq & p(s_i) \\
[\![\neg p]\!]_k^i & \triangleq & \neg p(s_i) \\
[\![f \wedge g]\!]_k^i & \triangleq & [\![f]\!]_k^i \wedge [\![g]\!]_k^i \\
[\![f \vee g]\!]_k^i & \triangleq & [\![f]\!]_k^i \vee [\![g]\!]_k^i \\
[\![\Box f]\!]_k^i & \triangleq & false \\
[\![\Diamond f]\!]_k^i & \triangleq & \bigvee_{j=i}^{k} [\![f]\!]_k^j \\
[\![\bigcirc f]\!]_k^i & \triangleq & \text{if } i < k \text{ then } [\![f]\!]_k^{i+1} \text{ else } false \\
[\![f \,\mathcal{U}\, g]\!]_k^i & \triangleq & \bigvee_{j=i}^{k} ([\![g]\!]_k^j \wedge \bigwedge_{n=i}^{j-1} [\![f]\!]_k^n) \\
[\![f \,\mathcal{R}\, g]\!]_k^i & \triangleq & \bigvee_{j=i}^{k} ([\![f]\!]_k^j \wedge \bigwedge_{n=i}^{j} [\![g]\!]_k^n)
\end{array}
$$

Trans. of an LTL formula for a Loop

- For an LTL formula f and $k, l, i \in \mathbb{N}$, with $l, i \leq k$,

$$
\begin{array}{lcl}
{}_l[\![p]\!]_k^i & \triangleq & p(s_i) \\
{}_l[\![\neg p]\!]_k^i & \triangleq & \neg p(s_i) \\
{}_l[\![f \wedge g]\!]_k^i & \triangleq & {}_l[\![f]\!]_k^i \wedge {}_l[\![g]\!]_k^i \\
{}_l[\![f \vee g]\!]_k^i & \triangleq & {}_l[\![f]\!]_k^i \vee {}_l[\![g]\!]_k^i
\end{array}
$$

Trans. of an LTL formula for a Loop

$$
\begin{aligned}
{}_l[\![\Box f]\!]_k^i &\triangleq \bigwedge_{j=\min(i,l)}^{k} {}_l[\![f]\!]_k^j \\
{}_l[\![\Diamond f]\!]_k^i &\triangleq \bigvee_{j=\min(i,l)}^{k} {}_l[\![f]\!]_k^j \\
{}_l[\![\bigcirc f]\!]_k^i &\triangleq {}_l[\![f]\!]_k^{succ(i)}
\end{aligned}
$$

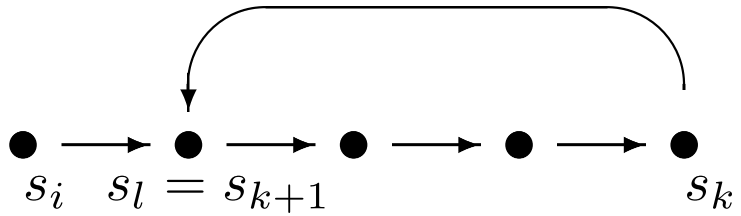
- Let $k, l, i \in \mathbb{N}$, with $l, i \leq k$.

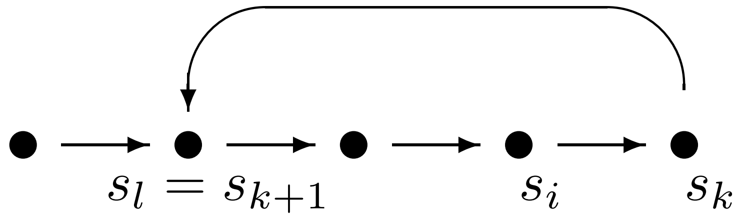
$$
succ(i) \triangleq \begin{cases} i+1 & \text{for } i < k \\ l & \text{for } i = k \end{cases}
$$

Trans. of an LTL formula for a Loop (cont.)

$$
\begin{aligned}
{}_l[\![f\ \mathcal{U}\ g]\!]_k^i \ \triangleq\ & \bigvee_{j=i}^{k}({}_l[\![g]\!]_k^j \wedge \bigwedge_{n=i}^{j-1} {}_l[\![f]\!]_k^n) \vee \\
& \bigvee_{j=l}^{i-1}({}_l[\![g]\!]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[\![f]\!]_k^n \wedge \bigwedge_{n=l}^{j-1} {}_l[\![f]\!]_k^n)
\end{aligned}
$$

$$
\begin{aligned}
{}_l[\![f\ \mathcal{R}\ g]\!]_k^i \ \triangleq\ & \bigwedge_{j=\min(i,l)}^{k} {}_l[\![g]\!]_k^j \vee \\
& \bigvee_{j=i}^{k}({}_l[\![f]\!]_k^j \wedge \bigwedge_{n=i}^{j} {}_l[\![g]\!]_k^n) \vee \\
& \bigvee_{j=l}^{i-1}({}_l[\![f]\!]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[\![g]\!]_k^n \wedge \bigwedge_{n=l}^{j} {}_l[\![g]\!]_k^n)
\end{aligned}
$$





Loop Condition

- The loop condition L_k is used to distinguish paths with bound k which are loops or not loops.
- For $k, l \in \mathbb{N}$, let
  - ${}_lL_k \triangleq T(s_k, s_l)$
  - $L_k \triangleq \bigvee_{l=0}^{k} {}_lL_k$.

General Translation

- Let f be an LTL formula, M a Kripke structure, and $k \in \mathbb{N}$.

$$[\![M, f]\!]_k \triangleq [\![M]\!]_k \wedge ((\neg L_k \wedge [\![f]\!]_k^0) \vee (\bigvee_{l=0}^{k} ({}_lL_k \wedge {}_l[\![f]\!]_k^0)))$$

  Note: is the term $\neg L_k$ redundant?

- **Theorem 4** $[\![M, f]\!]_k$ *is satisfiable iff* $M \models_k$ ***E*** f*.*
- **Corollary 5** $M \models$ ***A*** $\neg f$ *iff* $[\![M, f]\!]_k$ *is unsatisfiable for all* $k \in \mathbb{N}$*.*

Bounds for LTL

- LTL model checking is known to be PSPACE-complete.
- A polynomial bound on k with respect to the size of M and f for which $M \models_k \mathbf{E}\ f \Leftrightarrow M \models \mathbf{E}\ f$ is unlikely to be found.
- **Theorem 6** *Given an LTL formula f and a Kripke structure M, let $|M|$ be the number of states in M, then $M \models \mathbf{E}\ f$ iff there exists $k \leq |M| \times 2^{|f|}$ with $M \models_k \mathbf{E}\ f$.*
- For the subset of LTL formulae that involves only temporal operators $\diamond$ and $\Box$, LTL model checking is NP-complete.
- For this subset of LTL formulae, there exists a bound on k linear in the number of states and the size of the formula.

Bounds for LTL (cont.)

- **Definition 7 (Loop Diameter)** *A Kripke structure is lasso shaped if every path p starting from an initial state is of the form $u_p v_p^\omega$, where u_p and v_p are finite sequences of length less or equal to u and v, respectively. The loop diameter of M is defined as (u, v).*
- **Theorem 8** *Given an LTL formula f and a lasso-shaped Kripke structure M, let the loop diameter of M be (u, v), then $M \models$ **E** f iff there exists $k \leq u + v$ with $M \models_k$ **E** f.*

Part II:

Bounded Model Checking for LTL with Past

Note: (k, l)-loop here corresponds to $(k-1, l)$-loop in Part I. For easy cross-referencing with the original paper, we have not attempted to unify the notion.

Propositional Temporal Logic

- The full propositional temporal logic (PTL) is LTL with past operators.

$$
\begin{array}{lll}
(\pi, i) \models \ominus f & \text{iff} & i > 0 \text{ and } (\pi, i-1) \models f \\
(\pi, i) \models \tilde{\ominus} f & \text{iff} & i = 0 \text{ or } (\pi, i-1) \models f \\
(\pi, i) \models \diamondminus f & \text{iff} & \exists j, j \leq i.(\pi, j) \models f \\
(\pi, i) \models \boxminus f & \text{iff} & \forall j, j \leq i.(\pi, j) \models f \\
(\pi, i) \models f \; \mathcal{S} \; g & \text{iff} & \exists j, j \leq i.((\pi, j) \models g \text{ and } \forall k, j < k \leq i.(\pi, k) \models f) \\
(\pi, i) \models f \; \mathcal{T} \; g & \text{iff} & \forall j, j \leq i.((\pi, j) \models g \text{ or } \exists k, j < k \leq i.(\pi, k) \models f)
\end{array}
$$

- Every PTL formula can be converted into the negation normal form.

Extend the Translation without Loops

- Let $k, i \in \mathbb{N}$ with $i \leq k$.

$$
\begin{array}{lcl}
[\![\ominus f]\!]_k^i & \triangleq & \begin{cases} false & i = 0 \\ [\![f]\!]_k^{i-1} & i > 0 \end{cases} \\
[\![\circleddash\!\!\sim f]\!]_k^i & \triangleq & \begin{cases} true & i = 0 \\ [\![f]\!]_k^{i-1} & i > 0 \end{cases} \\
[\![\diamondminus f]\!]_k^i & \triangleq & \bigvee_{j=0}^{i} [\![f]\!]_k^j \\
[\![\boxminus f]\!]_k^i & \triangleq & \bigwedge_{j=0}^{i} [\![f]\!]_k^j \\
[\![f \; \mathcal{S} \; g]\!]_k^i & \triangleq & \bigvee_{j=0}^{i} ([\![g]\!]_k^j \wedge \bigwedge_{n=j+1}^{i} [\![f]\!]_k^n) \\
[\![f \; \mathcal{T} \; g]\!]_k^i & \triangleq & \bigwedge_{j=0}^{i} ([\![g]\!]_k^j \vee \bigvee_{n=j+1}^{i} [\![f]\!]_k^n)
\end{array}
$$

Extend the Translation with Loops

- The extension is not straightforward.
- For example, consider the path $01(2345)^\omega$ which can be seen as a $(6,2)$-loop.
  - In the future case, the encoding of a specification is based on the idea that, for every time in the encoding, exactly one successor time exists.
  - Past formulae do not enjoy the above property.
    - The predecessor of 2 may be 1 or 5.

$x =$	0	1	2	3	4	5	2	3	4	5	2	3	4	5	2
time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

The Solution: Intuition

- The formula $\Diamond(x = 2 \wedge \diamondminus(x = 3 \wedge \diamondminus(x = 4 \wedge (\diamondminus(x = 5)))))$ is true in all the occurrences of $x = 2$ after the fourth.
- The key idea is that every formula has a finite discriminating power for events in the past.
- When evaluated sufficiently far from the origin of time, a formula becomes unable to distinguish its past sequence from infinitely many other past sequences with a "similar" behavior.
- The idea is then to collapse the undistinguishable versions of the past together into the same equivalence class.

Past Temporal Horizon

- The past temporal horizon (PTH) $\tau_\pi(f)$ of a PTL formula f with respect to a (k,l)-loop π (with period $p = k - l$) is the smallest value $n \in \mathbb{N}$ such that

$$\forall i, l \leq i < k.((\pi, i + np) \models f \text{ iff } (\forall n' > n.(\pi, i + n'p) \models f)).$$

PTH of a PTL Formula

- The PTH $\tau(f)$ of a PTL formula f is defined as $\tau(f) \triangleq \max_{\tau \in \Pi} \tau_\pi(f)$ where Π is the set of all the paths which are (k, l)-loops for some $k > l \geq 0$.
- **Theorem 9** *Let f and g be PTL formulae. Then, it holds that:*
  - $\tau(p) = 0$, *when* $p \in A$ *and* $\tau(f) = \tau(\neg f)$*;*
  - $\tau(\circ f) \leq \tau(f)$, *when* $\circ \in \{\bigcirc, \Diamond, \Box\}$*;*
  - $\tau(\circ f) \leq \tau(f) + 1$, *when* $\circ \in \{\ominus, \tilde{\bigcirc}, \diamondminus, \boxminus\}$*;*
  - $\tau(f \circ g) \leq \max(\tau(f), \tau(g))$, *when* $\circ \in \{\wedge, \vee, \mathcal{U}, \mathcal{R}\}$*;*
  - $\tau(f \circ g) \leq \max(\tau(f), \tau(g)) + 1$, *when* $\circ \in \{\mathcal{S}, \mathcal{T}\}$*;*
- The PTH of a PTL formula is bounded by its structure regardless of the particular path π.

Borders and Intervals

- We call
  - **LB**$(n) \triangleq l + np$ the n-th left border of π,
  - **RB**$(n) \triangleq k + np$ the n-th right border of π, and
  - the interval *M*$(n) \triangleq [0, \textbf{RB}(n))$ the n-th main domain of a (k, l)-loop.
- We call
  - **LB**$(f) \triangleq \textbf{LB}(\tau(f))$ the left border of f,
  - **RB**$(f) \triangleq \textbf{RB}(\tau(f))$ the right border of f, and
  - *M*$(f) \triangleq M(\tau(f))$ the main domain of f.

Borders and Intervals (cont.)

$x =$	0	1	2	3	4	5	2	3	4	5	2	3	4	5	2
time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- $\textbf{LB}(0) = 2$
- $\textbf{RB}(0) = 6$
- $\textbf{LB}(1) = 6$
- $\textbf{RB}(1) = 10$

Projection of Points

- Let $i \in \mathbb{N}$.
- The projection of the point i in the n-th main domain of a (k,l)-loop is $\rho_n(i)$, defined as

$$\rho_n(i) \triangleq \begin{cases} i & i < \textbf{RB}(n) \\ \rho_n(i-p) & \text{otherwise} \end{cases}$$

- The projection of the point i onto the main domain of f is defined as $\rho_f(i) \triangleq \rho_{\tau(f)}(i)$.

Projection of Intervals

- The projection of the interval $[a, b)$ onto the main domain of f is defined as $\rho_f([a, b)) \triangleq \rho_{\tau(f)}([a, b))$.
- **Lemma 10** *For an open interval* $[a, b)$*,*

$$\rho_n([a,b)) = \begin{cases} \emptyset & \textit{if } a = b, \textit{ else} \\ [a,b) & \textit{if } b < \textbf{\textit{RB}}(n), \textit{ else} \\ [\min(a, \textbf{\textit{LB}}(n)), \textbf{\textit{RB}}(n)) & \textit{if } b - a \geq p, \textit{ else} \\ [\rho_n(a), \rho_n(b)) & \textit{if } \rho_n(a) < \rho_n(b), \textit{ else} \\ [\rho_n(a), \textbf{\textit{RB}}(n)) \cup [\textbf{\textit{LB}}(n), \rho_n(b)) & \end{cases}$$

Extended Projection of Intervals

- An extended intervals is of the form $[a, b)$ where b is possibly less than a (or even it is equal to ∞).
- Let $[a, b)$ be an extended interval.
- The extended projection of $[a, b)$ onto the n-th main domain of a (k, l)-loop is defined as follows

$$\rho_n^*([a,b)) \triangleq \begin{cases} \rho_n^*([a, \max(a, \mathbf{RB}(n)) + p)) & b = \infty \\ \rho_n^*([a, b + p)) & b < a \\ \rho_n^*([a, b)) & \text{otherwise} \end{cases}$$

- As before, $\rho_f^*([a,b)) \triangleq \rho_{\tau(f)}^*([a,b))$.

Equivalent Counterparts

- **Theorem 11** *For*
  - *any PTL formula* f*,*
  - *any* (k, l)*-loop* π*, and*
  - *any extended interval* $[a, b)$*,*

  a point $i \in [a, b)$ *such that* $(\pi, i) \models f$ *exists iff a point* $i' \in \rho_f^*([a, b))$ *exists such that* $(\pi, i') \models f$*.*

Extend the Translation with Loops

- The translation of a PTL formula on a (k, l)-loop π at time point i (with $k, l, i \in \mathbb{N}$ and $0 \leq l < k$) is a propositional formula inductively defined as follows.

$$
\begin{array}{lcl}
{}_l[\![p]\!]_k^i & \triangleq & p^{\rho_0(i)} \\
{}_l[\![\neq p]\!]_k^i & \triangleq & \neq p^{\rho_0(i)} \\
{}_l[\![f \wedge g]\!]_k^i & \triangleq & {}_l[\![f]\!]_k^{\rho_f(i)} \wedge {}_l[\![g]\!]_k^{\rho_g(i)} \\
{}_l[\![f \vee g]\!]_k^i & \triangleq & {}_l[\![f]\!]_k^{\rho_f(i)} \vee {}_l[\![g]\!]_k^{\rho_g(i)}
\end{array}
$$

Extend the Translation with Loops (cont.)

$$
\begin{array}{lcl}
{}_l[\![\Diamond f]\!]_k^i & \triangleq & \bigvee_{j\in\rho_f^*([i,\infty))} {}_l[\![f]\!]_k^j \\
{}_l[\![\Box f]\!]_k^i & \triangleq & \bigwedge_{j\in\rho_f^*([i,\infty))} {}_l[\![f]\!]_k^j \\
{}_l[\![f\ \mathcal{U}\ g]\!]_k^i & \triangleq & \bigvee_{j\in\rho_g^*([i,\infty))} ({}_l[\![g]\!]_k^j \wedge \bigwedge_{n\in\rho_f^*([i,j))} {}_l[\![f]\!]_k^n) \\
{}_l[\![f\ \mathcal{R}\ g]\!]_k^i & \triangleq & \bigwedge_{j\in\rho_g^*([i,\infty))} ({}_l[\![g]\!]_k^j \vee \bigvee_{n\in\rho_f^*([i,j))} {}_l[\![f]\!]_k^n) \\
{}_l[\![\ominus f]\!]_k^i & \triangleq & i > 0 \wedge {}_l[\![f]\!]_k^{\rho_f(i-1)} \\
{}_l[\![\circleddash f]\!]_k^i & \triangleq & i = 0 \vee {}_l[\![f]\!]_k^{\rho_f(i-1)} \\
{}_l[\![\diamondminus f]\!]_k^i & \triangleq & \bigvee_{j\in\rho_f^*([0,i])} {}_l[\![f]\!]_k^j \\
{}_l[\![\boxminus f]\!]_k^i & \triangleq & \bigwedge_{j\in\rho_f^*([0,i])} {}_l[\![f]\!]_k^j \\
{}_l[\![f\ \mathcal{S}\ g]\!]_k^i & \triangleq & \bigvee_{j\in\rho_g^*([0,i])} ({}_l[\![g]\!]_k^j \wedge \bigwedge_{n\in\rho_f^*((j,i])} {}_l[\![f]\!]_k^n) \\
{}_l[\![f\ \mathcal{T}\ g]\!]_k^i & \triangleq & \bigwedge_{j\in\rho_g^*([0,i])} ({}_l[\![g]\!]_k^j \vee \bigvee_{n\in\rho_f^*((j,i])} {}_l[\![f]\!]_k^n)
\end{array}
$$

Correctness of the Translation

- **Theorem 12** *For any PTL formula f, a (k,l)-loop path π in M such that $\pi \models f$ exists iff $[\![M]\!]_k \wedge {}_lL_k \wedge {}_l[\![f]\!]_k^0$ is satisfiable.*