Program Verification: Hoare Logic

(Based on [Apt and Olderog 1991; Gries 1981; Hoare 1969; Kleymann 1999; Sethi 1996])

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An Axiomatic View of Programs

- The properties of a program can, in principle, be found out from its text by means of purely *deductive reasoning*.
- The deductive reasoning involves the application of valid *inference rules* to a set of valid *axioms*.
- The choice of axioms will depend on the choice of programming languages.
- We shall introduce such an axiomatic approach, called the *Hoare logic*, to program correctness.

Assertions

- When executed, a program will evolve through different *states*, which are essentially a mapping of the program variables to values in their respective domains.
- To reason about correctness of a program, we inevitably need to talk about its states.
- An *assertion* is a precise statement about the state of a program.
- Most interesting assertions can be expressed in a *first-order* language.

Pre and Post-conditions

- The behavior of a "structured" (single-entry/single-exit) program statement can be characterized by attaching assertions at the entry and the exit of the statement.
- For a statement S, this is conveniently expressed as a so-called *Hoare triple*, denoted $\{P\}\ S\ \{Q\}$, where
  - P is called the *pre-condition* and
  - Q is called the *post-condition* of S.

Interpretations of a Hoare Triple

- A Hoare triple $\{P\}\ S\ \{Q\}$ may be interpreted in two different ways:
  - **Partial Correctness**: if the execution of S starts in a state satisfying P and terminates, then it results in a state satisfying Q.
  - **Total Correctness**: if the execution of S starts in a state satisfying P, then it will terminate and result in a state satisfying Q.

Note: sometimes we write $\langle P\rangle\ S\ \langle Q\rangle$ when total correctness is intended.

Pre and Post-Conditions for Specification

- Find an integer approximate to the square root of another integer n:

  $$\{0 \leq n\} \; ? \; \{d^2 \leq n < (d+1)^2\}$$

  or slightly better (clearer about what can be changed)

  $$\{0 \leq n\} \; d := ? \; \{d^2 \leq n < (d+1)^2\}$$

- Find the index of value x in an array b:
  - $\{x \in b[0..n-1]\} \; ? \; \{0 \leq i < n \wedge x = b[i]\}$
  - $\{0 \leq n\} \; ? \; \{(0 \leq i < n \wedge x = b[i]) \vee (i = n \wedge x \notin b[0..n-1])\}$

Note: there are other ways to stipulate which variables are to be changed and which are not.

A Little Bit of History

The following seminal paper started it all:

> *C.A.R. Hoare. An axiomatic basis for computer programs. CACM, 12(8):576-580, 1969.*

- Original notation: $P\ \{S\}\ Q$ (vs. $\{P\}\ S\ \{Q\}$)
- Interpretation: partial correctness
- Provided axioms and proof rules

Note: R.W. Floyd did something similar for flowcharts earlier in 1967, which was also a precursor of "proof outline" (a program fully annotated with assertions).

The Assignment Statement

- Syntax:

$$x := E$$

- Meaning: execution of the assignment $x := E$ (read as "x becomes E") evaluates E and stores the result in variable x.
- We will assume that expression E in $x := E$ has *no side-effect* (i.e., does not change the value of any variable).
- Which of the following two Hoare triples is correct about the assignment $x := E$?
  - $\{P\}\ x := E\ \{P[E/x]\}$
  - $\{Q[E/x]\}\ x := E\ \{Q\}$

  Note: E is essentially a first-order term.

Some Hoare Triples for Assignments

- $\{x > 0\}\ x := x - 1\ \{x \geq 0\}$
  or equivalently, $\{x - 1 \geq 0\}\ x := x - 1\ \{x \geq 0\}$
- $\{x + 1 > 5\}\ x := x + 1\ \{x > 5\}$
- $\{5 \neq 5\}\ x := 5\ \{x \neq 5\}$

Axiom of the Assignment Statement

$$\frac{}{\{Q[E/x]\}\ x := E\ \{Q\}}\ (\mathit{Assignment})$$

Why is this so?

- Let s be the state before $x := E$ and s' the state after.
- So, $s' = s[x := E]$ assuming E has no side-effect.
- $Q[E/x]$ holds in s if and only if Q holds in s', because
  - every variable, except x, in $Q[E/x]$ and Q has the same value in s and s', and
  - $Q[E/x]$ has every x in Q replaced by E, while Q has every x evaluated to E in s' ($= s[x := E]$).

The Multiple Assignment Statement

- Syntax:

$$x_1, x_2, \cdots, x_n := E_1, E_2, \cdots, E_n$$

  where x_i's are distinct variables.
- Meaning: execution of the multiple assignment evaluates all E_i's and stores the results in the corresponding variables x_i's.
- Examples:
  - $i, j := 0, 0$ (initialize i and j to 0)
  - $x, y := y, x$ (swap x and y)
  - $g, p := g + 1, p - 1$ (increment g by 1, while decrement p by 1)
  - $i, x := i + 1, x + i$ (increment i by 1 and x by i)

Some Hoare Triples for Multi-assignments

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- Swapping two values
  $\{x < y\}\ x, y := y, x\ \{y < x\}$
- Number of games in a tournament
  $\{g + p = n\}\ g, p := g + 1, p - 1\ \{g + p = n\}$
- Taking a sum
  $\{x + i = 1 + 2 + \cdots + (i + 1 - 1)\}$
  $i, x := i + 1, x + i$
  $\{x = 1 + 2 + \cdots + (i - 1)\}$

Simultaneous Substitution

- $P[E/x]$ can be naturally extended to allow E to be a list $E_1, E_2, \cdots, E_n$ and x to be $x_1, x_2, \cdots, x_n$, all of which are distinct variables.
- $P[E/x]$ is then the result of simultaneously replaying $x_1, x_2, \cdots, x_n$ with the corresponding expressions $E_1, E_2, \cdots, E_n$; enclose E_i's in parentheses if necessary.
- Examples:
  - $(x < y)[y, x/x, y] = (y < x)$
  - $(g + p = n)[g + 1, p - 1/g, p] = ((g + 1) + (p - 1) = n) = (g + p = n)$
  - $(x = 1 + 2 + \cdots + (i - 1))[i + 1, x + i/i, x]$ $= ((x + i) = 1 + 2 + \cdots + ((i + 1) - 1))$ $= (x + i = 1 + 2 + \cdots + ((i + 1) - 1))$

Axiom of the Multiple Assignment

- Syntax:

$$x_1, x_2, \cdots, x_n := E_1, E_2, \cdots, E_n$$

  where x_i's are distinct variables.
- Axiom:

$$\frac{}{\{Q[E_1, \cdots, E_n/x_1, \cdots, x_n]\}\ x_1, \cdots, x_n := E_1, \cdots, E_n\ \{Q\}}\ (Assign.)$$

Assignment to an Array Entry

- Syntax:

$$b[i] := E$$

- Notation for an altered array: $(b; i : E)$ denotes the array that is identical to b, except that entry i stores the value of E.

$$(b; i : E)[j] = \begin{cases} E & \text{if } i = j \\ b[j] & \text{if } i \neq j \end{cases}$$

- Axiom:

$$\frac{}{\{Q[(b; i : E)/b]\}\ b[i] := E\ \{Q\}}\ (\mathit{Assignment})$$

Pre and Post-condition of a Loop

- A precondition just before a loop can capture the conditions for executing the loop.
- An assertion just within a loop body can capture the conditions for staying in the loop.
- A postcondition just after a loop can capture the conditions upon leaving the loop.

A Simple Example

$\{x \geq 0 \land y > 0\}$
while $x \geq y$ **do**
$\quad \{x \geq 0 \land y > 0 \land x \geq y\}$
$\quad x := x - y$
od
$\{x \geq 0 \land y > 0 \land x \not\geq y\}$
// or
$\{x \geq 0 \land y > 0 \land x < y\}$

More about the Example

We can say more about the program.

// may assume $x, y := m, n$ here for some $m \geq 0$ and $n > 0$
$\{x \geq 0 \land y > 0 \land (x \equiv m \pmod{y})\}$
while $x \geq y$ **do**
    $x := x - y$
od
$\{x \geq 0 \land y > 0 \land (x \equiv m \pmod{y}) \land x < y\}$

Note: repeated subtraction is a way to implement the integer division. So, the program is taking the residue of x divided by y.

A Simple Programming Language

- To study inference rules of Hoare logic, we consider a simple programming language with the following syntax for statements:

$$
\begin{array}{rcl}
S & ::= & \textbf{skip} \\
 & | & x := E \\
 & | & S_1; S_2 \\
 & | & \textbf{if } B \textbf{ then } S \textbf{ fi} \\
 & | & \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi} \\
 & | & \textbf{while } B \textbf{ do } S \textbf{ od}
\end{array}
$$

Proof Rules

$$\frac{}{\{Q[E/x]\}\ x := E\ \{Q\}} \quad \text{(Assignment)}$$

$$\frac{}{\{P\}\ \textbf{skip}\ \{P\}} \quad \text{(Skip)}$$

$$\frac{\{P\}\ S_1\ \{Q\} \qquad \{Q\}\ S_2\ \{R\}}{\{P\}\ S_1; S_2\ \{R\}} \quad \text{(Sequence)}$$

$$\frac{\{P \wedge B\}\ S_1\ \{Q\} \qquad \{P \wedge \neg B\}\ S_2\ \{Q\}}{\{P\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{Q\}} \quad \text{(Conditional)}$$

"**if** B **then** S **fi**" can be treated as "**if** B **then** S **else skip fi**" or directly with the following rule:

$$\frac{\{P \wedge B\}\ S\ \{Q\} \qquad P \wedge \neg B \to Q}{\{P\}\ \textbf{if}\ B\ \textbf{then}\ S\ \textbf{fi}\ \{Q\}} \quad \text{(If-Then)}$$

Proof Rules (cont.)

$$\frac{\{P \wedge B\}\ S\ \{P\}}{\{P\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{P \wedge \neg B\}} \qquad \text{(While)}$$

$$\frac{P \rightarrow P' \qquad \{P'\}\ S\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ S\ \{Q\}} \qquad \text{(Consequence)}$$

Note: with a suitable notion of validity, the set of proof rules up to now can be shown to be sound and (relatively) complete for programs that use only the considered constructs.

Some Auxiliary Rules

$$\frac{P \rightarrow P' \qquad \{P'\}\ S\ \{Q\}}{\{P\}\ S\ \{Q\}} \qquad \text{(Strengthening Precondition)}$$

$$\frac{\{P\}\ S\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ S\ \{Q\}} \qquad \text{(Weakening Postcondition)}$$

$$\frac{\{P_1\}\ S\ \{Q_1\} \qquad \{P_2\}\ S\ \{Q_2\}}{\{P_1 \wedge P_2\}\ S\ \{Q_1 \wedge Q_2\}} \qquad \text{(Conjunction)}$$

$$\frac{\{P_1\}\ S\ \{Q_1\} \qquad \{P_2\}\ S\ \{Q_2\}}{\{P_1 \vee P_2\}\ S\ \{Q_1 \vee Q_2\}} \qquad \text{(Disjunction)}$$

Note: these rules provide more convenience, but do not actually add deductive power.

Invariants

- An *invariant* at some point of a program is an assertion that holds whenever execution of the program reaches that point.
- Assertion P in the rule for a while loop is called a *loop invariant* of the while loop.
- An assertion is called an *invariant of an operation* (a segment of code) if, assumed true before execution of the operation, the assertion remains true after execution of the operation.
- Invariants are a bridge between the static text of a program and its dynamic computation.

Program Annotation

- Inserting assertions/invariants in a program as comments helps understanding of the program.

  $\{x \geq 0 \land y > 0 \land (x \equiv m \pmod{y})\}$
  while $x \geq y$ **do**
      $\{x \geq 0 \land y > 0 \land x \geq y \land (x \equiv m \pmod{y})\}$
      $x := x - y$
      $\{y > 0 \land x \geq 0 \land (x \equiv m \pmod{y})\}$
  od
  $\{x \geq 0 \land y > 0 \land (x \equiv m \pmod{y}) \land x < y\}$

- A correct annotation of a program can be seen as a partial proof outline for the program.
- Boolean assertions can also be used as an aid to program testing.

An Annotated Program

$\{x \geq 0 \land y \geq 0 \land gcd(x,y) = gcd(m,n)\}$
while $x \neq 0$ and $y \neq 0$ do
    $\{x \geq 0 \land y \geq 0 \land gcd(x,y) = gcd(m,n)\}$
    if $x < y$ then $x, y := y, x$ fi;
    $\{x \geq y \land y \geq 0 \land gcd(x,y) = gcd(m,n)\}$
    $x := x - y$
    $\{x \geq 0 \land y \geq 0 \land gcd(x,y) = gcd(m,n)\}$
od
$\{(x = 0 \land y \geq 0 \land y = gcd(x,y) = gcd(m,n)) \lor$
$(x \geq 0 \land y = 0 \land x = gcd(x,y) = gcd(m,n))\}$

Note: m and n are two arbitrary non-negative integers, at least one of which is nonzero.

Total Correctness: Termination

- All inference rules introduced so far, except the **while** rule, work for total correctness.
- Below is a rule for the total correctness of the **while** statement:

$$\frac{\{P \wedge B\}\ S\ \{P\} \qquad \{P \wedge B \wedge t = Z\}\ S\ \{t < Z\} \qquad P \rightarrow (t \geq 0)}{\{P\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{P \wedge \neg B\}}$$

  where t is an integer-valued expression (state function) and Z is a "rigid" variable that does not occur in P, B, t, or S.
- The above function t is called a *rank* (or variant) function.

Termination of a Simple Program

$g, p := 0, n; \quad // \; n \geq 1$
while $p \geq 2$ **do**
$\quad g, p := g + 1, p - 1$
od

- Loop Invariant: $(g + p = n) \wedge (p \geq 1)$
- Rank (Variant) Function: p
- The loop terminates when $p = 1$ $(p \geq 1 \wedge p \not\geq 2)$.

Well-Founded Sets

- A binary relation $\preceq \subseteq A \times A$ is a **partial order** if it is
  - reflexive: $\forall x \in A(x \preceq x)$,
  - transitive: $\forall x, y, z \in A((x \preceq y \land y \preceq z) \rightarrow x \preceq z)$, and
  - antisymmetric: $\forall x, y \in A((x \preceq y \land y \preceq x) \rightarrow x = y)$.
- A partially ordered set $(W, \preceq)$ is **well-founded** if there is no infinite decreasing chain $x_1 \succ x_2 \succ x_3 \succ \cdots$ of elements from W. (Note: "$x \succ y$" means "$y \preceq x \land y \neq x$".)
- Examples:
  - $(Z_{\geq 0}, \leq)$
  - $(Z_{\geq 0} \times Z_{\geq 0}, \leq)$,
    where $(x_1, y_1) \leq (x_2, y_2)$ if $(x_1 < x_2) \lor (x_1 = x_2 \land y_1 \leq y_2)$

Termination by Well-Founded Induction

Below is a more general rule for the total correctness of the **while** statement:

$$\frac{\{P \wedge B\}\ S\ \{P\} \qquad \{P \wedge B \wedge \delta = D\}\ S\ \{\delta \prec D\} \qquad P \rightarrow (\delta \in W)}{\{P\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{P \wedge \neg B\}}$$

where $(W, \preceq)$ is a well-founded set, δ is a state function, and D is a "rigid" variable ranged over W that does not occur in P, B, δ, or S.

Nondeterminism

- Syntax of the Alternative Statement:

  if $B_1 \rightarrow S_1$
  $[\!]\ B_2 \rightarrow S_2$
  $\ldots$
  $[\!]\ B_n \rightarrow S_n$
  fi

  Each of the "$B_i \rightarrow S_i$"s is called a guarded command, where B_i is the guard of the command and S_i the body.
- Semantic:
  1. One of the guarded commands, whose guard evaluates to true, is nondeterministically selected and its body executed.
  2. If none of the guards evaluates to true, then the execution aborts.

Rule for the Alternative Statement

- The Alternative Statement:

  $$\begin{array}{l} \textbf{if } B_1 \rightarrow S_1 \\ \ [\!] \ B_2 \rightarrow S_2 \\ \ldots \\ \ [\!] \ B_n \rightarrow S_n \\ \textbf{fi} \end{array}$$

- Inference rule:

$$\frac{P \rightarrow B_1 \vee \cdots \vee B_n \qquad \{P \wedge B_i\}\ S_i\ \{Q\}, \text{ for } 1 \leq i \leq n}{\{P\}\ \textbf{if } B_1 \rightarrow S_1 [\!] \cdots [\!]\ B_n \rightarrow S_n \textbf{ fi}\ \{Q\}}$$

The Coffee Can Problem as a Program

$$
\begin{array}{l}
B, W := m, n; \quad // \; m > 0 \land n > 0 \\
\textbf{while } B + W \geq 2 \textbf{ do} \\
\quad \textbf{if } B \geq 0 \land W > 1 \rightarrow B, W := B + 1, W - 2 \quad // \text{ same color} \\
\quad\; [\!] \; B > 1 \land W \geq 0 \rightarrow B, W := B - 1, W \quad // \text{ same color} \\
\quad\; [\!] \; B > 0 \land W > 0 \rightarrow B, W := B - 1, W \quad // \text{ different colors} \\
\quad \textbf{fi} \\
\textbf{od}
\end{array}
$$

- Loop Invariant: $W \equiv n \pmod 2$ $\quad$ (and $B + W \geq 1$)
- Variant (Rank) Function: $B + W$
- The loop terminates when $B + W = 1$.

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