Homework Assignment #2

Note

This assignment is due 9:10AM Thursday, October 22, 2009. Please write or type your answers on A4 (or similar size) paper. Put your completed homework on the instructor's desk before the class starts. For late submissions, please drop them in Yih-Kuen Tsay's mail box on the first floor of Management College Building II. Late submissions will be penalized by 20% for each working day overdue. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. Prove, using Gentzen's System LK, the validity of the following sequents:
   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall x P(x) \rightarrow \forall x Q(x)$ (10 points)
   (b) $\vdash \exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$ (10 points)
   (c) $\exists x A(x) \rightarrow B \vdash \forall x(A(x) \rightarrow B)$, assuming x does not occur free in B. (10 points)
2. Prove, using *Natural Deduction*, the validity of the following sequents:
   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall x P(x) \rightarrow \forall x Q(x)$ (10 points)
   (b) $\vdash \exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$ (10 points)
   (c) $\forall x(A(x) \rightarrow B) \vdash \exists x A(x) \rightarrow B$, assuming x does not occur free in B. (10 points)
3. From the axioms for equality ($=$), prove using *Natural Deduction* that $=$ is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity). (20 points)
   (a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry)
   (b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity)
4. Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality ($=$). (20 points)
   (a) $\dfrac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}$ ($=$ *Symmetry*)
   (b) $\dfrac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}$ ($=$ *Transitivity*)