Homework Assignment #3

Note

This assignment is due 9:10AM Thursday, October 29, 2009. Please write or type your answers on A4 (or similar size) paper. Put your completed homework on the instructor's desk before the class starts. For late submissions, please drop them in Yih-Kuen Tsay's mail box on the first floor of Management Building II. A late submission will be penalized by 20% for each working day overdue. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. Please give a formal inductive definition for $A[t_1, t_2/x_1, x_2]$ (which denotes the formula obtained from simultaneously substituting t_1 and t_2 for free occurrences of x_1 and x_2 respectively in A) that we have explained informally in class. (20 points)

2. A first-order theory for *groups* contains the following three axioms:

   - $\forall a \forall b \forall c(a \cdot (b \cdot c) = (a \cdot b) \cdot c)$. (Associativity)
   - $\forall a((a \cdot e = a) \wedge (e \cdot a = a))$. (Identity)
   - $\forall a((a \cdot a^{-1} = e) \wedge (a^{-1} \cdot a = e))$. (Inverse)

   Here $\cdot$ is the binary operation, e is a constant, called the identity, and $(\cdot)^{-1}$ is the inverse function that returns the inverse of an element. Let M denote the set of the three axioms. Prove, using *Natural Deduction* plus the derived rules in HW#2, the validity of the following sequents:

   (a) $M \vdash \forall a \forall b \forall c((a \cdot b = a \cdot c) \rightarrow b = c)$. (Hint: a typical proof in algebra books is the following: $b = e \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c) = (a^{-1} \cdot a) \cdot c = e \cdot c = c$.) (40 points)

   (b) $M \vdash \forall a \forall b \forall c(((a \cdot b = e) \wedge (b \cdot a = e) \wedge (a \cdot c = e) \wedge (c \cdot a = e)) \rightarrow b = c)$, which says that every element has a unique inverse. (Hint: a typical proof in algebra books is the following: $b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c$.) (40 points)