

## The Sequent Calculus: System *LK*

- Axioms

$$\frac{}{A \vdash A} \text{ (Axiom)}$$

For convenience, you may extend this to the following:

$$\frac{}{\Gamma, A \vdash A, \Delta} \text{ (Axiom)}$$

- Logical Rules (I)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} (\wedge : \text{Left}_1) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} (\wedge : \text{Right})$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} (\wedge : \text{Left}_2)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} (\vee : \text{Left}) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} (\vee : \text{Right}_1)$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} (\vee : \text{Right}_2)$$

For convenience, you may take the following alternatives:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} (\wedge : \text{Left}) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} (\wedge : \text{Right})$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} (\vee : \text{Left}) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} (\vee : \text{Right})$$

- Logical Rules (II)

$$\frac{\Gamma \vdash A, \Delta_1 \quad \Gamma, B \vdash \Delta_2}{\Gamma, A \rightarrow B \vdash \Delta_1, \Delta_2} (\rightarrow : \text{Left}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} (\rightarrow : \text{Right})$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} (\neg : \text{Left}) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} (\neg : \text{Right})$$

- Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (Weakening : Left)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ (Weakening : Right)}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (Contraction : Left)} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ (Contraction : Right)}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta} \text{ (Exchange : Left)} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2} \text{ (Exchange : Right)}$$

For convenience, you may treat  $\Gamma, \Delta$ , etc. as *sets*,  $A, B$  as  $\{A\}, \{B\}$ , and the comma (in “ $\Gamma, A$ ” etc.) as set union, so that you can do without these structural rules; use this alternative with the extended notion of an axiom.

- The Cut Rule

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ (Cut)}$$

- Quantifier Rules

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \text{ (\forall : Left)} \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \text{ (\forall : Right)}$$

$$\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \text{ (\exists : Left)} \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \text{ (\exists : Right)}$$

In these quantifier rules, we assume that all substitutions are admissible,  $y$  is not free in  $A$ , and  $y$  does not occur free in the lower sequent.

Axioms for Equality (an extension for languages with =):

Let  $t, s_1, \dots, s_n, t_1, \dots, t_n$  be arbitrary terms.

$$\overline{\Gamma \vdash t = t, \Delta}$$

For every  $n$ -ary function  $f$ ,

$$\overline{\Gamma, s_1 = t_1, \dots, s_n = t_n \vdash f(s_1, \dots, s_n) = f(t_1, \dots, t_n), \Delta}$$

For every  $n$ -ary predicate  $P$  (including =),

$$\overline{\Gamma, s_1 = t_1, \dots, s_n = t_n, P(s_1, \dots, s_n) \vdash P(t_1, \dots, t_n), \Delta}$$