Homework Assignment #3

Note

This assignment contains several exercise problems for you to practice writing formal statements in first-order logic. It will not be graded and you are not required to turn in your work.

Problems

We assume the binding powers of the logical connectives decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$.

1. Consider the structure $\mathcal{N} = (\mathrm{N}, \{+, \times, 0, 1, <\})$, i.e., the set of natural numbers with the usual functions, constants, and predicates ("=" is implicitly assumed to be a binary predicate).

   (a) Write a first-order formula to define the set of odd numbers (i.e., a formula with a free variable such that the formula is true exactly when the free variable is assigned an odd number).

   (b) Write a first-order formula to define the set of prime numbers.

2. Consider the set of natural numbers with addition $(\mathrm{N}, \{+\})$ and the set of integers with addition $(\mathrm{Z}, \{+\})$. Give a first-order sentence that is true in one but false in the other. (Note: two structures are said to be *elementarily equivalent* if they satisfy the same set of first-order sentences. So, the sentence you would give shows that $(\mathrm{N}, \{+\})$ and $(\mathrm{Z}, \{+\})$ are not elementarily equivalent.)

3. Consider the set of integers with the $<$ relation $(\mathrm{Z}, \{<\})$ and the set of real numbers with the $<$ relation $(\mathrm{R}, \{<\})$. Give a first-order sentence that is true in one but false in the other, showing that $(\mathrm{Z}, \{<\})$ and $(\mathrm{R}, \{<\})$ are not elementarily equivalent. (Hint: discrete vs. dense sets.)