

# Propositional Logic

(Based on [Gallier 1986], [Goubault-Larrecq and Mackie 1997], and [Huth and Ryan 2004])

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# Introduction

- 🌐 Logic concerns two concepts:
  - ☀️ **truth** (in a specific or general context)
  - ☀️ **provability** (of truth from assumed truth)
- 🌐 *Formal (symbolic) logic* approaches logic by rules for manipulating symbols:
  - ☀️ **syntax** rules: for writing statements (or formulae).  
(There are also semantic rules determining whether a statement is true or false in a context or mathematical structure.)
  - ☀️ **inference** rules: for obtaining true statements from other true statements.
- 🌐 We shall introduce two main branches of formal logic:
  - ☀️ *propositional logic*
  - ☀️ *first-order logic* (predicate logic/calculus)
- 🌐 The following slides cover **propositional logic**.

# Propositions

- 🌐 A *proposition* is a statement that is either *true* or *false* such as the following:
  - ☀️ Leslie is a teacher.
  - ☀️ Leslie is rich.
  - ☀️ Leslie is a pop singer.
- 🌐 Simplest (*atomic*) propositions may be combined to form *compound* propositions:
  - ☀️ Leslie is *not* a teacher.
  - ☀️ *Either* Leslie is not a teacher *or* Leslie is not rich.
  - ☀️ *If* Leslie is a pop singer, *then* Leslie is rich.

# Inferences

- 🌐 We are given the following assumptions:
  - ☀️ Leslie is a teacher.
  - ☀️ Either Leslie is not a teacher or Leslie is not rich.
  - ☀️ If Leslie is a pop singer, then Leslie is rich.
- 🌐 We wish to conclude the following:
  - ☀️ Leslie is not a pop singer.
- 🌐 The above process is an example of *inference* (deduction). Is it correct?

# Symbolic Propositions




- Propositions are represented by *symbols*, when only their truth values are of concern.
  - $P$ : Leslie is a teacher.
  - $Q$ : Leslie is rich.
  - $R$ : Leslie is a pop singer.
- Compound propositions can then be more succinctly written.
  - not*  $P$ : Leslie is not a teacher.
  - not*  $P$  *or* *not*  $Q$ : Either Leslie is not a teacher or Leslie is not rich.
  - $R$  *implies*  $Q$ : If Leslie is a pop singer, then Leslie is rich.

# Symbolic Inferences







- 🌐 We are given the following assumptions:
  - ☀️  $P$  (Leslie is a teacher.)
  - ☀️  $\text{not } P \text{ or not } Q$  (Either Leslie is not a teacher or Leslie is not rich.)
  - ☀️  $R \text{ implies } Q$  (If Leslie is a pop singer, then Leslie is rich.)
- 🌐 We wish to conclude the following:
  - ☀️  $\text{not } R$  (Leslie is not a pop singer.)
- 🌐 Correctness of the inference may be checked by asking:
  - ☀️ Is  $(P \text{ and } (\text{not } P \text{ or not } Q) \text{ and } (R \text{ implies } Q)) \text{ implies } (\text{not } R)$  a tautology (valid formula)?
  - ☀️ Or, is  $(A \text{ and } (\text{not } A \text{ or not } B) \text{ and } (C \text{ implies } B)) \text{ implies } (\text{not } C)$  a tautology (valid formula)?

# Propositional Logic: Syntax

## Vocabulary:



-  A countable set  $\mathcal{P}$  of *proposition symbols* (variables):  $P, Q, R, \dots$  (also called *atomic propositions*);
-  *Logical connectives* (operators):  $\neg, \wedge, \vee, \rightarrow$ , and  $\leftrightarrow$  and sometimes the constant  $\perp$  (*false*);
-  Auxiliary symbols: “(”, “)”


## How to read the logical connectives.


-   $\neg$  (negation): not
-   $\wedge$  (conjunction): and
-   $\vee$  (disjunction): or
-   $\rightarrow$  (implication): implies (or if  $\dots$ , then  $\dots$ )
-   $\leftrightarrow$  (equivalence): is equivalent to (or if and only if)
-   $\perp$  (*false* or bottom): false (or bottom)





# Propositional Logic: Syntax (cont.)

## Propositional Formulae:

-  Any  $A \in \mathcal{P}$  is a formula and so is  $\perp$  (these are the “atomic” formula).
-  If  $A$  and  $B$  are formulae, then so are  $\neg A$ ,  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ , and  $(A \leftrightarrow B)$ .

  $A$  is called a *subformula* of  $\neg A$ , and  $A$  and  $B$  subformulae of  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ , and  $(A \leftrightarrow B)$ .

 Precedence (for avoiding excessive parentheses):

-   $A \wedge B \rightarrow C$  means  $((A \wedge B) \rightarrow C)$ .
-   $A \rightarrow B \vee C$  means  $(A \rightarrow (B \vee C))$ .
-   $A \rightarrow B \rightarrow C$  means  $(A \rightarrow (B \rightarrow C))$ .
-  More about this later ...



# About Boolean Expressions

🌐 *Boolean expressions* are essentially propositional formulae, though they may allow more things as atomic formulae.

🌐 Boolean expressions:

☀  $(x \vee y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y}) \wedge x$

☀  $(x + y + \bar{z}) \cdot (\bar{x} + \bar{y}) \cdot x$

☀  $(a \vee b \vee \bar{c}) \wedge (\bar{a} \vee \bar{b}) \wedge a$

☀ etc.

🌐 Propositional formula:  $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q) \wedge P$

# Propositional Logic: Semantics

- The meanings of propositional formulae may be conveniently summarized by the **truth table**:

$A$	$B$	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
$T$	$T$	$F$	$T$	$T$	$T$	$T$
$T$	$F$	$F$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$F$	$T$	$T$	$F$
$F$	$F$	$T$	$F$	$F$	$T$	$T$

The meaning of  $\perp$  is always  $F$  (false).

- There is an implicit inductive definition in the table. We shall try to make this precise.

# Truth Assignment and Valuation

- 🌐 The semantics of propositional logic assigns a truth function to each propositional formula.
- 🌐 Let  $BOOL$  be the set of truth values  $\{T, F\}$ .
- 🌐 A *truth assignment* (valuation) is a function from  $\mathcal{P}$  (the set of proposition symbols) to  $BOOL$ .
- 🌐 Let  $PROPS$  be the set of all propositional formulae.
- 🌐 A truth assignment  $v$  may be extended to a *valuation* function  $\hat{v}$  from  $PROPS$  to  $BOOL$  as follows:

# Truth Assignment and Valuation (cont.)

$$\hat{v}(\perp) = F$$

$$\hat{v}(P) = v(P) \text{ for all } P \in \mathcal{P}$$

$$\hat{v}(P) = \text{as defined by the table below, otherwise}$$

$\hat{v}(A)$	$\hat{v}(B)$	$\hat{v}(\neg A)$	$\hat{v}(A \wedge B)$	$\hat{v}(A \vee B)$	$\hat{v}(A \rightarrow B)$	$\hat{v}(A \leftrightarrow B)$
$T$	$T$	$F$	$T$	$T$	$T$	$T$
$T$	$F$	$F$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$F$	$T$	$T$	$F$
$F$	$F$	$T$	$F$	$F$	$T$	$T$

So, the truth value of a propositional formula is completely determined by the truth values of its subformulae.

# Truth Assignment and Satisfaction

- 🌐 We say  $v \models A$  ( $v$  *satisfies*  $A$ ) if  $\hat{v}(A) = T$ .
- 🌐 So, the symbol  $\models$  denotes a binary relation, called *satisfaction*, between truth assignments and propositional formulae.
- 🌐  $v \not\models A$  ( $v$  *falsifies*  $A$ ) if  $\hat{v}(A) = F$ .

Alternatively (in a more generally applicable format), the satisfaction relation  $\models$  may be defined as follows:

$$\begin{aligned} v &\not\models \perp \\ v &\models P &\iff v(P) = T, \quad \text{for all } P \in \mathcal{P} \\ v &\models \neg A &\iff v \not\models A \text{ (it is not the case that } v \models A) \\ v &\models A \wedge B &\iff v \models A \text{ and } v \models B \\ v &\models A \vee B &\iff v \models A \text{ or } v \models B \\ v &\models A \rightarrow B &\iff v \not\models A \text{ or } v \models B \\ v &\models A \leftrightarrow B &\iff (v \models A \text{ and } v \models B) \\ & &\text{or } (v \not\models A \text{ and } v \not\models B) \end{aligned}$$

# Object vs. Meta Language

- 🌐 The language that we study is referred to as the *object* language.
- 🌐 The language that we use to study the object language is referred to as the *meta* language.
- 🌐 For example, *not*, *and*, and *or* that we used to define the satisfaction relation  $\models$  are part of the meta language.

- 🌐 A proposition  $A$  is *satisfiable* if there exists an assignment  $v$  such that  $v \models A$ .
  - ☀️  $v(P) = F, v(Q) = T \models (P \vee Q) \wedge (\neg P \vee \neg Q)$
- 🌐 A proposition is *unsatisfiable* if no assignment satisfies it.
  - ☀️  $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge P$  is unsatisfiable.
- 🌐 The problem of determining whether a given proposition is satisfiable is called the *satisfiability problem*.



# Tautology and Validity

- 🌐 A proposition  $A$  is a *tautology* if every assignment satisfies  $A$ , written as  $\models A$ .
  - ☀️  $\models A \vee \neg A$
  - ☀️  $\models (A \wedge B) \rightarrow (A \vee B)$
- 🌐 The problem of determining whether a given proposition is a tautology is called the *tautology problem*.
- 🌐 A proposition is also said to be *valid* if it is a tautology.
- 🌐 So, the problem of determining whether a given proposition is valid (a tautology) is also called the *validity problem*.



Note: the notion of a tautology is restricted to propositional logic. In first-order logic, we also speak of valid formulae.

# Validity vs. Satisfiability

## Theorem

*A proposition  $A$  is valid (a tautology) if and only if  $\neg A$  is unsatisfiable.*

So, there are two ways of proving that a proposition  $A$  is a tautology:

-   $A$  is satisfied by every truth assignment (or  $A$  cannot be falsified by any truth assignment).
-   $\neg A$  is unsatisfiable.

# Relating the Logical Connectives

## Lemma

$$\models (A \leftrightarrow B) \leftrightarrow ((A \rightarrow B) \wedge (B \rightarrow A))$$

$$\models (A \rightarrow B) \leftrightarrow (\neg A \vee B)$$

$$\models (A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$$

$$\models \perp \leftrightarrow (A \wedge \neg A)$$

Note: these equivalences imply that some connectives could be dispensed with. We normally want a smaller set of connectives when analyzing properties of the logic and a larger set when actually using the logic.

# Normal Forms

- 🌐 A *literal* is an atomic proposition or its negation.
- 🌐 A propositional formula is in **Conjunctive Normal Form (CNF)** if it is a conjunction of disjunctions of literals.
  - ☀  $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q) \wedge P$
  - ☀  $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$
- 🌐 A propositional formula is in **Disjunctive Normal Form (DNF)** if it is a disjunction of conjunctions of literals.
  - ☀  $(P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q) \vee P$
  - ☀  $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R)$
- 🌐 A propositional formula is in **Negation Normal Form (NNF)** if negations occur only in literals.
  - ☀ CNF or DNF is also NNF (but not vice versa).
  - ☀  $(P \wedge \neg Q) \wedge (P \vee (Q \wedge \neg R))$  in NNF, but not CNF or DNF.
- 🌐 Every propositional formula has an equivalent formula in each of these normal forms.

# Semantic Entailment

- Consider two sets of propositions  $\Gamma$  and  $\Delta$ .
- We say that  $v \models \Gamma$  ( $v$  satisfies  $\Gamma$ ) if  $v \models B$  for every  $B \in \Gamma$ ; analogously for  $\Delta$ .
- We say that  $\Delta$  is a *semantic consequence* of  $\Gamma$  if every assignment that satisfies  $\Gamma$  also satisfies  $\Delta$ , written as  $\Gamma \models \Delta$ .
  - $A, A \rightarrow B \models A, B$
  - $A \rightarrow B, \neg B \models \neg A$
- We also say that  $\Gamma$  *semantically entails*  $\Delta$  when  $\Gamma \models \Delta$ .

# Sequents

- 🌐 A (**propositional**) *sequent* is an expression of the form  $\Gamma \vdash \Delta$ , where  $\Gamma = A_1, A_2, \dots, A_m$  and  $\Delta = B_1, B_2, \dots, B_n$  are finite (possibly empty) sequences of (**propositional**) formulae.
- 🌐 In a sequent  $\Gamma \vdash \Delta$ ,  $\Gamma$  is called the *antecedent* (also *context*) and  $\Delta$  the *consequent*.

Note: many authors prefer to write a sequent as  $\Gamma \longrightarrow \Delta$  or  $\Gamma \Longrightarrow \Delta$ , while reserving the symbol  $\vdash$  for provability (deducibility) in the proof (deduction) system under consideration.

## Sequents (cont.)

- 🌐 A sequent  $A_1, A_2, \dots, A_m \vdash B_1, B_2, \dots, B_n$  is **falsifiable** if there exists a valuation  $v$  such that
 
$$v \models (A_1 \wedge A_2 \wedge \dots \wedge A_m) \wedge (\neg B_1 \wedge \neg B_2 \wedge \dots \wedge \neg B_n).$$
  - ☀️  $A \vee B \vdash B$  is falsifiable, as
 
$$v(A) = T, v(B) = F \models (A \vee B) \wedge \neg B.$$
- 🌐 A sequent  $A_1, A_2, \dots, A_m \vdash B_1, B_2, \dots, B_n$  is **valid** if, for every valuation  $v$ ,  $v \models A_1 \wedge A_2 \wedge \dots \wedge A_m \rightarrow B_1 \vee B_2 \vee \dots \vee B_n$ .
  - ☀️  $A \vdash A, B$  is valid.
  - ☀️  $A, B \vdash A \wedge B$  is valid.
- 🌐 A sequent is **valid** if and only if it is **not falsifiable**.
- 🌐 In the following, we will use only sequents of this simpler form:
 
$$A_1, A_2, \dots, A_m \vdash C,$$
 where  $C$  is a formula.

# Inference Rules

- 🌐 Inference rules allow one to obtain true statements from other true statements.
- 🌐 Below is an inference rule for conjunction.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} (\wedge I)$$

- 🌐 In an inference rule, the upper sequents (above the horizontal line) are called the *premises* and the lower sequent is called the *conclusion*.



- 🌐 A **deduction tree** is a tree where each node is labeled with a sequent such that, for every internal (non-leaf) node,
  - ☀️ the label of the **node** corresponds to the **conclusion** and
  - ☀️ the labels of its **children** correspond to the **premises** of an instance of an inference rule.
- 🌐 A **proof tree** is a deduction tree, each of whose leaves is labeled with an axiom.
- 🌐 The root of a deduction or proof tree is called the **conclusion**.
- 🌐 A sequent is **provable** if there exists a proof tree of which it is the conclusion.

## Detour: Another Style of Proofs

- 🌐 Proofs may also be carried out in a calculational style (like in algebra); for example,

$$\begin{aligned}
 & (A \vee B) \rightarrow C \\
 \equiv & \{ A \rightarrow B \equiv \neg A \vee B \} \\
 & \neg(A \vee B) \vee C \\
 \equiv & \{ \text{de Morgan's law} \} \\
 & (\neg A \wedge \neg B) \vee C \\
 \equiv & \{ \text{distributive law} \} \\
 & (\neg A \vee C) \wedge (\neg B \vee C) \\
 \equiv & \{ A \rightarrow B \equiv \neg A \vee B \} \\
 & (A \rightarrow C) \wedge (B \rightarrow C) \\
 \Rightarrow & \{ A \wedge B \Rightarrow A \} \\
 & (A \rightarrow C)
 \end{aligned}$$

- 🌐 Here,  $\Rightarrow$  corresponds to semantical entailment and  $\equiv$  to mutual semantical entailment. Both are transitive.

🌐 Equivalence is **commutative** and **associative**

$$\odot A \leftrightarrow B \equiv B \leftrightarrow A$$

$$\odot A \leftrightarrow (B \leftrightarrow C) \equiv (A \leftrightarrow B) \leftrightarrow C$$

$$\🌐 \perp \vee A \equiv A \vee \perp \equiv A$$

$$\🌐 \neg A \wedge A \equiv \perp$$

$$\🌐 A \rightarrow B \equiv \neg A \vee B$$

$$\🌐 A \rightarrow \perp \equiv \neg A$$

$$\🌐 (A \vee B) \rightarrow C \equiv (A \rightarrow C) \wedge (B \rightarrow C)$$

$$\🌐 A \rightarrow (B \rightarrow C) \equiv (A \wedge B) \rightarrow C$$

$$\🌐 A \rightarrow B \equiv A \leftrightarrow (A \wedge B)$$

$$\🌐 A \wedge B \Rightarrow A$$

# Natural Deduction in the Sequent Form

$$\frac{}{\Gamma, A \vdash A} (Ax)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} (\wedge I)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} (\wedge E_1)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} (\wedge E_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} (\vee I_1)$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} (\vee I_2)$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} (\vee E)$$

# Natural Deduction (cont.)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} (\rightarrow I)$$

$$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} (\rightarrow E)$$

$$\frac{\Gamma, A \vdash B \wedge \neg B}{\Gamma \vdash \neg A} (\neg I)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash B} (\neg E)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} (\neg\neg I)$$

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} (\neg\neg E)$$

These inference rules collectively are called System *ND* (the propositional part).

# A Proof in Propositional ND

Below is a partial proof of the validity of  $P \wedge (\neg P \vee \neg Q) \wedge (R \rightarrow Q) \rightarrow \neg R$  in ND, where  $\gamma$  denotes  $P \wedge (\neg P \vee \neg Q) \wedge (R \rightarrow Q)$ .

$$\begin{array}{c}
 \frac{\frac{\frac{\vdots}{\gamma, R \vdash R \rightarrow Q}}{\gamma, R \vdash Q} \quad \frac{\gamma, R \vdash R}{\gamma, R \vdash R} (Ax)}{\gamma, R \vdash Q} (\rightarrow E) \quad \frac{\frac{\frac{\vdots}{\gamma, R, Q \vdash P \wedge \neg P}}{\gamma, R \vdash \neg Q} (\neg I)}{\gamma, R \vdash Q \wedge \neg Q} (\wedge I)}{\gamma, R \vdash Q \wedge \neg Q} (\neg I)}{\frac{P \wedge (\neg P \vee \neg Q) \wedge (R \rightarrow Q) \vdash \neg R}{\vdash P \wedge (\neg P \vee \neg Q) \wedge (R \rightarrow Q) \rightarrow \neg R} (\rightarrow I)}
 \end{array}$$

# Soundness and Completeness

## Theorem

System  $ND$  is *sound*, i.e., if a sequent  $\Gamma \vdash C$  is *provable* in  $ND$ , then  $\Gamma \vdash C$  is *valid*.

## Theorem

System  $ND$  is *complete*, i.e., if a sequent  $\Gamma \vdash C$  is *valid*, then  $\Gamma \vdash C$  is *provable* in  $ND$ .

A set  $\Gamma$  of propositions is **satisfiable** if some valuation satisfies every proposition in  $\Gamma$ . For example,  $\{A \vee B, \neg B\}$  is satisfiable.

## Theorem

*For any (possibly infinite) set  $\Gamma$  of propositions, if every finite non-empty subset of  $\Gamma$  is satisfiable then  $\Gamma$  is satisfiable.*

Proof hint: by contradiction and the completeness of *ND*.



# Consistency

- A set  $\Gamma$  of propositions is *consistent* if there exists some proposition  $B$  such that the sequent  $\Gamma \vdash B$  is not provable.
- Otherwise,  $\Gamma$  is *inconsistent*; e.g.,  $\{A, \neg(A \vee B)\}$  is inconsistent.

## Lemma

*For System ND, a set  $\Gamma$  of propositions is **inconsistent** if and only if there is some proposition  $A$  such that both  $\Gamma \vdash A$  and  $\Gamma \vdash \neg A$  are provable.*

## Theorem

*For System ND, a set  $\Gamma$  of propositions is **satisfiable** if and only if  $\Gamma$  is **consistent**.*