Homework Assignment #5

Due Time/Date

11:59PM Tuesday, May 31, 2022. Late submission will be penalized by 20% for each working day overdue.

How to Submit

Please use a word processor or scan hand-written answers to produce a single PDF file. Name your file according to this pattern: "r107250xx-hw5". Upload the PDF file to the NTU COOL course site for Software Development Methods 2022. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

This assignment contains several exercise problems for you to practice writing formal statements in first-order logic. We assume the binding powers of the logical connectives decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$ (so that you may avoid using some parentheses).

1. (20 points) Consider the structure $\mathcal{N} = (\mathrm{N}, \{+, \times, 0, 1, 2, <\})$, i.e., the set of natural numbers with the usual functions, constants (0, 1, and 2), and predicates ("=" is implicitly assumed to be a binary predicate).

   (a) Write a first-order formula to define the set of odd numbers (i.e., a formula with a free variable such that the formula is true exactly when the free variable is assigned an odd number).

   (b) Write a first-order formula to define the set of prime numbers.

2. (20 points) Consider the set of integers with the $<$ relation $(\mathrm{Z}, \{<\})$ and the set of real numbers with the $<$ relation $(\mathrm{R}, \{<\})$. Give a first-order sentence that is true in one but false in the other. Two structures are said to be *elementarily equivalent* if they satisfy the same set of first-order sentences. So, the sentence you would give shows that $(\mathrm{Z}, \{<\})$ and $(\mathrm{R}, \{<\})$ are not elementarily equivalent. (Hint: discrete vs. dense sets.)

3. (60 points) Please provide a precise description, using logical formulae, for each of the following requirements. The functions/constants and predicates you may use are: $+$, $\times$, 0, 1, 2, $<$, $=$, $\leq$, plus those introduced in the requirement statements. Make assumptions where you see necessary.

   (a) The array $A[0..N-1]$ (of integers) represents a max heap with $A[0]$ as the root.

(b) The array $A[0..N-1]$ (of integers) is cyclically sorted in an increasing order. (Note: $3, 4, 0, 1, 2$, for example, is a cyclically sorted list of integers.)