Suggested Solutions for Homework Assignment #1

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\wedge, \vee\}$, $\to$, $\leftrightarrow$, $\vdash$.

1. Prove, using Gentzen's System LK, the validity of the following sequents:

   (a) $\vdash (\neg p \vee q) \to (p \to q)$ (10 points)

   (b) $p \vee q \to r \vdash (p \to r) \wedge (q \to r)$ (10 points)

   (c) $\vdash (p \to (q \to r)) \to (p \wedge q \to r)$ (10 points)

   (d) $\vdash ((p \to q) \to p) \to p$ (10 points)

   Solution. In the following, the right label L stands for *Left* and R stands for *Right*. We use the set semantics in the proofs of the following sequents.

   (a)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p, q}\,\textit{Axiom}}{\neg p, p \vdash q}\,\neg : L \qquad \dfrac{}{q, p \vdash q}\,\textit{Axiom}}{\neg p \vee q, p \vdash q}\,\vee : L}{\neg p \vee q \vdash p \to q}\,\to : R}{\vdash (\neg p \vee q) \to (p \to q)}\,\to : R$$

   (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\textit{Axiom}}{p \vdash p \vee q}\,\vee : R_1 \qquad \dfrac{}{r, p \vdash r}\,\textit{Axiom}}{p \vee q \to r, p \vdash r}\,\to : L}{p \vee q \to r \vdash p \to r}\,\to : R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\textit{Axiom}}{q \vdash p \vee q}\,\vee : R_2 \qquad \dfrac{}{r, q \vdash r}\,\textit{Axiom}}{p \vee q \to r, q \vdash r}\,\to : L}{p \vee q \to r \vdash q \to r}\,\to : R}{p \vee q \to r \vdash (p \to r) \wedge (q \to r)}\,\wedge : R$$

   (c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, q \vdash p}\,\textit{Axiom} \qquad \dfrac{\dfrac{}{p, q \vdash q}\,\textit{Axiom} \qquad \dfrac{}{p, q, r \vdash r}\,\textit{Axiom}}{p, q, q \to r \vdash r}\,\to : L}{p \to (q \to r), p, q \vdash r}\,\to : L}{p \to (q \to r), p \wedge q \vdash r}\,\wedge : L}{p \to (q \to r) \vdash (p \wedge q) \to r}\,\to : R}{\vdash (p \to (q \to r)) \to ((p \wedge q) \to r)}\,\to : R$$

   (d)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p, q}\,\textit{Axiom}}{\vdash p \to q, p}\,\to : R \qquad \dfrac{}{p \vdash p}\,\textit{Axiom}}{(p \to q) \to p \vdash p}\,\to : L}{\vdash ((p \to q) \to p) \to p}\,\to : R$$

$\square$

2. We presented in class the $\wedge : \mathit{Left}_1$ and $\wedge : \mathit{Left}_2$ rules of Gentzen's System LK as follows.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge : \mathit{Left}_1) \qquad\qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge : \mathit{Left}_2)$$

We also suggested that, for convenience, the following $\wedge : \mathit{Left}$ rule can be used instead:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge : \mathit{Left})$$

Show that every proof using the $\wedge : \mathit{Left}_1$ or $\wedge : \mathit{Left}_2$ rules may be converted into one using the $\wedge : \mathit{Left}$ rule and vice versa. (20 points)

Solution. In the following, the set semantics is used.

- Convert the proof using the $\wedge : \mathit{Left}_1$ rule into one using $\wedge : \mathit{Left}$ rule.

$$\text{Original:}\quad \dfrac{\dfrac{\dfrac{\cdots}{\Sigma, A, \vdash \Delta}}{\Sigma, A \wedge B \vdash \Delta}\,\wedge \mathit{Left}_1}{\cdots} \qquad \text{Converted:}\quad \dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\Sigma, A \vdash \Delta}}{\Sigma, A, B \vdash \Delta}\,\mathit{Weakening} : \mathit{Left}}{\Sigma, A \wedge B \vdash \Delta}\,\wedge : \mathit{Left}}{\cdots}$$

  Convert the proof using the $\wedge : \mathit{Left}_2$ rule into one using $\wedge : \mathit{Left}$ rule.

$$\text{Original:}\quad \dfrac{\dfrac{\dfrac{\cdots}{\Sigma, B, \vdash \Delta}}{\Sigma, A \wedge B \vdash \Delta}\,\wedge \mathit{Left}_2}{\cdots} \qquad \text{Converted:}\quad \dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\Sigma, B \vdash \Delta}}{\Sigma, A, B \vdash \Delta}\,\mathit{Weakening} : \mathit{Left}}{\Sigma, A \wedge B \vdash \Delta}\,\wedge : \mathit{Left}}{\cdots}$$

- Convert the proof using the $\wedge : \mathit{Left}$ rule into one using $\wedge : \mathit{Left}_1$ and $\wedge : \mathit{Left}_2$ rules.

$$\text{Original:}\quad \dfrac{\dfrac{\dfrac{\cdots}{\Sigma, A, B, \vdash \Delta}}{\Sigma, A \wedge B \vdash \Delta}\,\wedge \mathit{Left}}{\cdots} \qquad \text{Converted:}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\Sigma, A, B \vdash \Delta}}{\Sigma, A, A \wedge B \vdash \Delta}\,\wedge : \mathit{Left}_2}{\Sigma, A \wedge B, A \wedge B \vdash \Delta}\,\wedge : \mathit{Left}_1}{\Sigma, A \wedge B \vdash \Delta}\,\mathit{Contraction} : \mathit{Left}}{\cdots}$$

$\square$

3. Prove, using *Natural Deduction*, the validity of the following sequents:

   (a) $\vdash (p \to q) \to (\neg p \vee q)$ (10 points)

   (b) $(p \to r) \wedge (q \to r) \vdash p \vee q \to r$ (10 points)

   (c) $\vdash (p \wedge q \to r) \to (p \to (q \to r))$ (10 points)

   (d) $\vdash ((p \to q) \to p) \to p$ (10 points)

Solution.

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{}{p \to q, \neg(\neg p \lor q), p \vdash \neg(\neg p \lor q)}\,Hyp}{p \to q, \neg(\neg p \lor q), p \vdash (\neg p \lor q) \land \neg(\neg p \lor q)}\,\land I}{p \to q, \neg(\neg p \lor q) \vdash \neg p}\,\neg I}{p \to q, \neg(\neg p \lor q) \vdash \neg p \lor q}\,\lor I_1 \qquad \dfrac{}{p \to q, \neg(\neg p \lor q) \vdash \neg(\neg p \lor q)}\,Hyp}{p \to q, \neg(\neg p \lor q) \vdash (\neg p \lor q) \land \neg(\neg p \lor q)}\,\land I}{p \to q \vdash \neg\neg(\neg p \lor q)}\,\neg I}{p \to q \vdash \neg p \lor q}\,\neg\neg E}{\vdash (p \to q) \to (\neg p \lor q)}\,\to I$$

α :

$$\dfrac{\dfrac{\dfrac{}{p \to q, \neg(\neg p \lor q), p \vdash p \to q}\,Hyp \qquad \dfrac{}{p \to q, \neg(\neg p \lor q), p \vdash p}\,Hyp}{p \to q, \neg(\neg p \lor q), p \vdash q}\,\to E}{p \to q, \neg(\neg p \lor q), p \vdash \neg p \lor q}\,\lor I_2$$

(b)

$$\dfrac{\dfrac{\dfrac{}{(p \to r) \land (q \to r), p \lor q \vdash p \lor q}\,Hyp \qquad \alpha \qquad \beta}{(p \to r) \land (q \to r), p \lor q \vdash r}\,\lor E}{(p \to r) \land (q \to r) \vdash p \lor q \to r}\,\to I$$

α :

$$\dfrac{\dfrac{\dfrac{}{(p \to r) \land (q \to r), p \lor q, p \vdash (p \to r) \land (q \to r)}\,Hyp}{(p \to r) \land (q \to r), p \lor q, p \vdash p \to r}\,\land E_1 \qquad \dfrac{}{(p \to r) \land (q \to r), p \lor q, p \vdash p}\,Hyp}{(p \to r) \land (q \to r), p \lor q, p \vdash r}\,\to E$$

β :

$$\dfrac{\dfrac{\dfrac{}{(p \to r) \land (q \to r), p \lor q, q \vdash (p \to r) \land (q \to r)}\,Hyp}{(p \to r) \land (q \to r), p \lor q, q \vdash q \to r}\,\land E_2 \qquad \dfrac{}{(p \to r) \land (q \to r), p \lor q, q \vdash q}\,Hyp}{(p \to r) \land (q \to r), p \lor q, q \vdash r}\,\to E$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \land q \to r, p, q \vdash p \land q \to r}\,Hyp \qquad \dfrac{\dfrac{}{p \land q \to r, p, q \vdash p}\,Hyp \qquad \dfrac{}{p \land q \to r, p, q \vdash q}\,Hyp}{p \land q \to r, p, q \vdash p \land q}\,\land I}{p \land q \to r, p, q \vdash r}\,\to E}{p \land q \to r, p \vdash q \to r}\,\to I}{p \land q \to r \vdash p \to (q \to r)}\,\to I}{\vdash (p \land q \to r) \to (p \to (q \to r))}\,\to I$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p \vdash (p \to q) \to p}\,Hyp \qquad \alpha}{(p \to q) \to p, \neg p \vdash p}\to E \qquad \dfrac{}{(p \to q) \to p, \neg p \vdash \neg p}\,Hyp}{(p \to q) \to p, \neg p \vdash p \wedge \neg p}\wedge I}{(p \to q) \to p \vdash \neg\neg p}\neg I}{(p \to q) \to p \vdash p}\neg\neg E}{\vdash ((p \to q) \to p) \to p}\to I$$

α :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p, p, \neg q \vdash p}\,Hyp \qquad \dfrac{}{(p \to q) \to p, \neg p, p, \neg q \vdash \neg p}\,Hyp}{(p \to q) \to p, \neg p, p, \neg q \vdash p \wedge \neg p}\wedge I}{(p \to q) \to p, \neg p, p \vdash \neg\neg q}\neg I}{(p \to q) \to p, \neg p, p \vdash q}\neg\neg E}{(p \to q) \to p, \neg p \vdash p \to q}\to I$$

$\square$