Propositional Logic

(Based on [Gallier 1986], [Goubault-Larrecq and Mackie 1997], and [Huth and Ryan 2004])

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Introduction

- Logic concerns two concepts: truth and provability (of truth from assumed truth).
- *Formal (symbolic) logic* approaches logic by rules for manipulating symbols:
  - Syntax rules: for writing statements (or formulae).
  - Inference rules: for obtaining true statements from other true statements.
- We shall introduce two main branches of formal logic: *propositional logic* and *first-order logic*.
- The following slides cover propositional logic.

Propositions

- A *proposition* is a statement that is either *true* or *false* such as the following:
  - Leslie is a teacher.
  - Leslie is rich.
  - Leslie is a pop singer.
- Simplest (atomic) propositions may be combined to form compound propositions:
  - Leslie is *not* a teacher.
  - *Either* Leslie is not a teacher *or* Leslie is not rich.
  - *If* Leslie is a pop singer, *then* Leslie is rich.

Inferences

- We are given the following assumptions:
  - Leslie is a teacher.
  - Either Leslie is not a teacher or Leslie is not rich.
  - If Leslie is a pop singer, then Leslie is rich.
- We wish to conclude the following:
  - Leslie is not a pop singer.
- The above process is an example of *inference* (deduction). Is it correct?

Symbolic Propositions

- Propositions are represented by *symbols*, when only their truth values are of concern.
  - P: Leslie is a teacher.
  - Q: Leslie is rich.
  - R: Leslie is a pop singer.
- Compound propositions can then be more succinctly written.
  - $not\ P$: Leslie is not a teacher.
  - $not\ P\ or\ not\ Q$: Either Leslie is not a teacher or Leslie is not rich.
  - $R\ implies\ Q$: If Leslie is a pop singer, then Leslie is rich.

Symbolic Inferences

- We are given the following assumptions:
  - P (Leslie is a teacher.)
  - $not\ P\ or\ not\ Q$ (Either Leslie is not a teacher or Leslie is not rich.)
  - $R\ implies\ Q$ (If Leslie is a pop singer, then Leslie is rich.)
- We wish to conclude the following:
  - $not\ R$ (Leslie is not a pop singer.)
- Correctness of the inference may be checked by asking:
  - Is $(P\ and\ (not\ P\ or\ not\ Q)\ and\ (R\ implies\ Q))\ implies\ (not\ R)$ a tautology (valid formula)?
  - Or, is $(A\ and\ (not\ A\ or\ not\ B)\ and\ (C\ implies\ B))\ implies\ (not\ C)$ a tautology (valid formula)?

Propositional Logic: Syntax

- Vocabulary:
  - A countable set $\mathcal{P}$ of *proposition symbols* (variables): $P, Q, R, \ldots$ (also called *atomic propositions*);
  - *Logical connectives* (operators): $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ and sometimes the constant $\bot$ ($false$);
  - Auxiliary symbols: "(", ")".
- *Propositional Formulae*:
  - Any $A \in \mathcal{P}$ is a formula (and so is $\bot$).
  - If A and B are formulae, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.

Propositional Logic: Semantics

- The meanings of positional formulae may be conveniently summarized by the truth table:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

  The meaning of $\perp$ is always F (false).

- There is an implicit inductive definition in the table. We shall try to make this precise.

Truth Assignment and Valuation

- The semantics of propositional logic assigns a truth function to each propositional formula.
- Let $BOOL$ be the set of truth values $\{T, F\}$.
- A *truth assignment* (valuation) is a function from $\mathcal{P}$ (the set of proposition symbols) to $BOOL$.
- Let $PROPS$ be the set of all propositional formulae.
- A truth assignment v may be extended to a *valuation* function $\hat{v}$ from $PROPS$ to $BOOL$ as follows:

Truth Assignment and Valuation (cont.)

$$\hat{v}(\bot) = F$$

$$\hat{v}(P) = v(P) \text{ for all } P \in \mathcal{P}$$

$$\hat{v}(P) = \text{as defined by the table below, otherwise}$$

$\hat{v}(A)$	$\hat{v}(B)$	$\hat{v}(\neg A)$	$\hat{v}(A \wedge B)$	$\hat{v}(A \vee B)$	$\hat{v}(A \rightarrow B)$	$\hat{v}(A \leftrightarrow B)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Truth Assignment and Satisfaction

- We say $v \models A$ (v *satisfies* A) if $\hat{v}(A) = T$ and $v \not\models A$ (v *falsifies* A) if $\hat{v}(A) = F$.
- Alternatively, $\models$ may be defined as follows:

$$
\begin{array}{lcl}
v \not\models \bot & & \\
v \models P & \iff & v(P) = T, \quad \text{for all } P \in \mathcal{P} \\
v \models \neg A & \iff & v \not\models A \text{ (it is } not \text{ the case that } v \models A) \\
v \models A \wedge B & \iff & v \models A \; and \; v \models B \\
v \models A \vee B & \iff & v \models A \; or \; v \models B \\
v \models A \rightarrow B & \iff & v \not\models A \; or \; v \models B \\
v \models A \leftrightarrow B & \iff & (v \models A \; and \; v \models B) \\
 & & or \; (v \not\models A \; and \; v \not\models B)
\end{array}
$$

Object vs. Meta Language

- The language that we study is referred to as the *object* language.
- The language that we use to study the object language is referred to as the *meta* language.
- For example, not, and, and or that we used to define the satisfaction relation $\models$ are part of the meta language.

Satisfiability

- A proposition A is *satisfiable* if there exists an assignment v such that $v \models A$.
  - $v(P) = F, v(Q) = T \models (P \vee Q) \wedge (\neg P \vee \neg Q)$
- A proposition is *unsatisfiable* if no assignment satisfies it.
  - $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge P$ is unsatisfiable.
- The problem of determining whether a given proposition is satisfiable is called the *satisfiability problem*.

Tautology and Validity

- A proposition A is a *tautology* if every assignment satisfies A, written as $\models A$.
  - $\models A \vee \neg A$
  - $\models (A \wedge B) \rightarrow (A \vee B)$
- The problem of determining whether a given proposition is a tautology is called the *tautology problem*.
- A proposition is also said to be *valid* if it is a tautology.
- So, the problem of determining whether a given proposition is valid (a tautology) is also called the *validity problem*.

Note: The notion of a tautology is restricted to propositional logic. In first-order logic, we also speak of valid formulae.

Validity vs. Satisfiability

Theorem.
A proposition A is *valid* (a tautology) *if and only if* $\neg A$ is *unsatisfiable*.

So, there are two ways of proving that a proposition A is a tautology:

- A is satisfied by every truth assignment (or A *cannot be falsified* by any truth assignment).
- $\neg A$ is unsatisfiable.

Semantic Entailment

- Consider two sets of propositions Γ and Δ.
- We say that $v \models \Gamma$ (v satisfies Γ) if $v \models B$ for every $B \in \Gamma$; analogously for Δ.
- We say that Δ is a *semantic consequence* of Γ if every assignment that satisfies Γ also satisfies Δ, written as $\Gamma \models \Delta$.
  - $A, A \to B \models A, B$
  - $A \to B, \neg B \models \neg A$
- We also say that Γ *semantically entails* Δ when $\Gamma \models \Delta$.

Relating the Logical Connectives

Lemma.

$$\models (A \leftrightarrow B) \leftrightarrow ((A \rightarrow B) \wedge (B \rightarrow A))$$

$$\models (A \rightarrow B) \leftrightarrow (\neg A \vee B)$$

$$\models (A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$$

$$\models \bot \leftrightarrow (A \wedge \neg A)$$

Note: These equivalences imply that some connectives could be dispensed with. We normally want a smaller set of connectives when analyzing properties of the logic and a larger set when actually using the logic.

Normal Forms

- A *literal* is an atomic proposition or its negation.
- A propositional formula is in Conjunctive Normal Form (CNF) if it is a conjunction of disjunctions of literals.
  - $(P \lor Q \lor \neg R) \land (\neg P \lor \neg Q) \land P$
  - $(P \lor Q \lor \neg R) \land (\neg P \lor \neg Q \lor R) \land (P \lor \neg Q \lor \neg R)$
- A propositional formula is in Disjunctive Normal Form (DNF) if it is a disjunction of conjunctions of literals.
  - $(P \land Q \land \neg R) \lor (\neg P \land \neg Q) \lor P$
  - $(\neg P \land \neg Q \land R) \lor (P \land Q \land \neg R) \lor (\neg P \land Q \land R)$
- A propositional formula is in Negation Normal Form (NNF) if negations occur only in literals.
  - CNF or DNF is also NNF (but not vice versa).
  - $(P \land \neg Q) \land (P \lor (Q \land \neg R))$ in NNF, but not CNF or DNF.

Falsification: Search for Counter Examples

To prove that "$(A \wedge (\neg A \vee \neg B) \wedge (C \to B)) \to \neg C$" is a tautology, we may try to find a valuation that falsifies it.

In the attempt of falsification, we consider pairs of the form (Γ, Δ), where Γ is a list of propositions we try to make true and Δ a list of propositions we try to make false.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{similar to the right branch}}{(\langle A, \neg A, C \to B\rangle, \langle \neg C\rangle)} \qquad \dfrac{(\langle A, \neg B, B\rangle, \langle \neg C\rangle) \qquad (\langle A, \neg B\rangle, \langle \neg C, C\rangle)}{(\langle A, \neg B, C \to B\rangle, \langle \neg C\rangle)}}{(\langle A, \neg A \vee \neg B, C \to B\rangle, \langle \neg C\rangle)}}{(\langle A, (\neg A \vee \neg B) \wedge (C \to B)\rangle, \langle \neg C\rangle)}}{(\langle A \wedge (\neg A \vee \neg B) \wedge (C \to B)\rangle, \langle \neg C\rangle)}}{(\langle\rangle, \langle (A \wedge (\neg A \vee \neg B) \wedge (C \to B)) \to \neg C\rangle)}$$

Note: read the above from bottom to top.

Sequents

- A (propositional) *sequent* is an expression of the form $\Gamma \vdash \Delta$, where $\Gamma = A_1, A_2, \cdots, A_m$ and $\Delta = B_1, B_2, \cdots, B_n$ are finite (possibly empty) sequences of (propositional) formulae.
- In a sequent $\Gamma \vdash \Delta$, Γ is called the *antecedent* (also *context*) and Δ the *consequent*

Note: Many authors prefer to write a sequent as $\Gamma \longrightarrow \Delta$ or $\Gamma \Longrightarrow \Delta$, while reserving the symbol $\vdash$ for provability (deducibility) in the proof (deduction) system under consideration.

Sequents (cont.)

- A sequent $A_1, A_2, \cdots, A_m \vdash B_1, B_2, \cdots, B_n$ is falsifiable if there exists a valuation v such that $v \models (A_1 \wedge A_2 \wedge \cdots \wedge A_m) \wedge (\neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_n)$.
  - $A \vee B \vdash B$ is falsifiable, as $v(A) = T, v(B) = F \models (A \vee B) \wedge \neg B$.
- A sequent $A_1, A_2, \cdots, A_m \vdash B_1, B_2, \cdots, B_n$ is valid if, for every valuation v, $v \models A_1 \wedge A_2 \wedge \cdots \wedge A_m \rightarrow B_1 \vee B_2 \vee \cdots \vee B_n$.
  - $A \vdash A, B$ is valid.
  - $A, B \vdash A \wedge B$ is valid.
- A sequent is valid if and only if it is not falsifiable.

The Sequent Calculus: Logical Rules (I)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L_1)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L_2)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R_1)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L)$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R_2)$$

In an inference rule, the one or two upper sequents (above the horizontal line) are called the *premises* and the lower sequent is called the *conclusion*.

The Sequent Calculus: Logical Rules (I')

Some authors have taken the following alternatives:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} (\wedge L) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} (\wedge R)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} (\vee L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} (\vee R)$$

The Sequent Calculus: Logical Rules (II)

$$\frac{\Gamma \vdash A, \Delta_1 \qquad \Gamma, B \vdash \Delta_2}{\Gamma, A \to B \vdash \Delta_1, \Delta_2} (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} (\neg R)$$

The Sequent Calculus: Axioms

$$\overline{A \vdash A}$$

All sequents of the form $A \vdash A$ are immediately provable. It is convenient to extend this to the following:

$$\overline{\Gamma, A \vdash A, \Delta}$$

In other words, $\Gamma \vdash \Delta$ is an axiom if Γ and Δ contain some common proposition.

Note: For a sequent $\Gamma \vdash \Delta$ that is an axiom, it is not possible to make all propositions in Γ true and all propositions in Δ false.

The Sequent Calculus: Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(WL) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,(WR)$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(CL) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,(CR)$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\,(EL) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\,(ER)$$

Note: If we treat Γ, Δ, etc. as sets, A, B as $\{A\}$, $\{B\}$, and the comma (in "Γ, A" etc.) as set union, then we can do without these rules, but will need the extended notion of an axiom.

Proofs

- A deduction tree is a tree where each node is labeled with a sequent such that, for every internal (non-leaf) node, the label of the node and those of its children correspond respectively to the conclusion and the premises of an instance of an inference rule.
- A proof tree is a deduction tree, each of whose leaves is labeled with an axiom.
- The root of a deduction or proof tree is called the conclusion.
- A sequent is provable if there exists a proof tree of which it is the conclusion.

The Sequent Calculus: The Cut Rule

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (Cut)$$

Note: The cut rule has a very special status. Its usage, though not essential as far as completeness is concerned (from the "cut elimination" theorem), often results in much shorter proofs.

Soundness and Completeness

The preceding structural rules, logical rules, and axioms constitute the propositional part LK_0 of Gentzen's System LK.

Theorem.
System LK_0 is *sound*, i.e., if a sequent $\Gamma \vdash \Delta$ is provable in LK_0, then $\Gamma \vdash \Delta$ is valid.

Theorem.
System LK_0 is *complete*, i.e., if a sequent $\Gamma \vdash \Delta$ is valid, then $\Gamma \vdash \Delta$ is provable in LK_0.

Compactness

A set Γ of propositions is satisfiable if some valuation satisfies every proposition in Γ. For example, $\{A \vee B, \neg B\}$ is satisfiable.

Theorem.
For any (possibly infinite) set Γ of propositions, if *every finite non-empty subset* of Γ is satisfiable then Γ is satisfiable.

Proof hint: by contradiction and the completeness of LK.

Consistency

- A set Γ of propositions is *consistent* if there exists some proposition B such that the sequent $\Gamma \vdash B$ is not provable.
- Otherwise, Γ is *inconsistent*; e.g., $\{A, \neg(A \vee B)\}$ is inconsistent.

Lemma.
For System LK_0, a set Γ of propositions is inconsistent *if and only if* there is some proposition A such that both $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ are provable.

Theorem.
For System LK_0, a set Γ of propositions is satisfiable *if and only if* Γ is *consistent*.

Natural Deduction in the Sequent Form

$$\frac{}{\Gamma, A \vdash A}\ (Ax)$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge I)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge E_1)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge E_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee I_1)$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee I_2)$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee E)$$

Natural Deduction (cont.)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to I) \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\,(\to E)$$

$$\frac{\Gamma, A \vdash B \wedge \neg B}{\Gamma \vdash \neg A}\,(\neg I) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash \neg A}{\Gamma \vdash B}\,(\neg E)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}\,(\neg\neg I) \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\,(\neg\neg E)$$