Hoare Logic (II): Procedures

(Based on [Gries 1981; Slonneger and Kurtz 1995])

Yih-Kuen Tsay

(with help from Ming-Hsien Tsai)

Dept. of Information Management

National Taiwan University

Non-recursive Procedures

- We first consider procedures with *call-by-value* parameters (and *global* variables).
- Syntax:

$$\textbf{proc}\ \mathrm{p}(\textbf{in}\ x);\ S$$

  where x may be a list of variables, S does not contain p, and S does not change x.
- Inference rule:

$$\frac{\{P\}\ S\ \{Q\}}{\{P[a/x] \wedge I\}\ \mathrm{p}(a)\ \{Q[a/x] \wedge I\}}$$

  where a may not be a global variable changed by S and I does not refer to variables changed by S.

How It May Go Wrong

- Example: $\textbf{proc}\ \mathrm{p}(\textbf{in}\ x);\ b := 2x;$
- Below is an incorrect usage of the rule

$$\frac{\{x = 1\}\ b := 2x\ \{b = 2 \wedge x = 1\}}{\{(x = 1)[b/x]\}\ \mathrm{p}(b)\ \{(b = 2 \wedge x = 1)[b/x]\}}$$

since the conclusion is not valid

$$\{b = 1\}\ \mathrm{p}(b)\ \{b = 2 \wedge b = 1\}.$$

- The inference rule cannot be applied, because the global variable b is changed by procedure p.
- The problem is that x becomes an alias of b in the invocation $\mathrm{p}(b)$, while $\{x = 1\}\ b := 2x\ \{b = 2 \wedge x = 1\}$ does not take this into account.

Non-recursive Procedures (cont.)

- We now consider procedures with *call-by-value*, *call-by-value-result*, and *call-by-result* parameters.
- Syntax:

$$\mathbf{proc}\ \mathrm{p}(\mathbf{in}\ x;\ \mathbf{in\ out}\ y;\ \mathbf{out}\ z);\ S$$

where x, y, z may be lists of variables, S does not contain p, and and S does not change x.

- Inference rule:

$$\frac{\{P\}\ S\ \{Q\}}{\{P[a,b/x,y] \wedge I)\}\ \mathrm{p}(a,b,c)\ \{Q[b,c/y,z] \wedge I\}}$$

where b, c are (lists of) distinct variables, a, b, c may not be global variables changed by S, and I does not refer to variables changed by S.

Non-recursive Procedures (cont.)

- Using wp, one can justify the rule with the understanding that "$\mathrm{p}(a, b, c)$" is equivalent to "$x, y := a, b; S; b, c := y, z$".

Recursive Procedures

- A rule for recursive procedures without parameters:

$$\frac{\{P\}\ \mathrm{p}()\ \{Q\} \vdash \{P\}\ S\ \{Q\}}{\vdash \{P\}\ \mathrm{p}()\ \{Q\}}$$

  where p is defined as "$\mathbf{proc}\ \mathrm{p}();\ S$".

- A rule for recursive procedures with parameters:

$$\frac{\forall v(\{P[v/x]\}\ \mathrm{p}(v)\ \{Q[v/x]\}) \vdash \{P\}\ S\ \{Q\}}{\vdash \{P[a/x]\}\ \mathrm{p}(a)\ \{Q[a/x]\}}$$

  where p is defined as "$\mathbf{proc}\ \mathrm{p}(\mathbf{in}\ x);\ S$" and a may not be a global variable changed by S.

An Example

$$
\begin{array}{l}
\textbf{proc } \mathrm{nonzero}(); \\
\textbf{begin} \\
\quad \textbf{read } x; \\
\quad \textbf{if } x = 0 \textbf{ then } \mathrm{nonzero}() \textbf{ fi}; \\
\textbf{end}
\end{array}
$$

- The semantics of "**read** x" is defined as follows:

$$\{IN = v \cdot L \wedge P[v/x]\} \ \textbf{read } x \ \{IN = L \wedge P\}$$

where v is a single value and L is a stream of values.

- We wish to prove the following:

$\{IN = Z \cdot n \cdot L \wedge$ "Z contains only zeros" $\wedge\, n \neq 0\}$ // $\{P\}$
$\mathrm{nonzero}();$
$\{IN = L \wedge x = n \wedge n \neq 0\}$ // $\{Q\}$

An Example (cont.)

- It amounts to proving the following annotation:

  proc nonzero();
  begin
  $\{IN = Z \cdot n \cdot L \wedge$ "Z contains only zeros" $\wedge\, n \neq 0\}$ // $\{P\}$
  read x;
  if $x = 0$ **then** nonzero() **fi**;
  $\{IN = L \wedge x = n \wedge n \neq 0\}$ // $\{Q\}$
  end

- The first step is to find a suitable assertion R between "**read** x" and the "**if**" statement.
- For this, we consider two cases: (1) Z is empty and (2) Z is not empty.

An Example (cont.)

- Case 1: Z is empty

  $\{IN = n \cdot L \wedge n \neq 0\}$

  read x

  $\{IN = L \wedge x = n \wedge n \neq 0\}$

- Case 2: Z is not empty

  $\{IN = 0 \cdot Z' \cdot n \cdot L \wedge$ "Z' contains only zeros" $\wedge\, n \neq 0\}$

  read x

  $\{IN = Z' \cdot n \cdot L \wedge$ "Z' contains only zeros" $\wedge\, n \neq 0 \wedge x = 0\}$

- Applying the Disjunction rule, we get a suitable R:

  $(IN = L \wedge x = n \wedge n \neq 0) \vee$

  $(IN = Z' \cdot n \cdot L \wedge$ "Z' contains only zeros" $\wedge\, n \neq 0 \wedge x = 0)$

An Example (cont.)

- We now have to prove the following:

$$\{R\}\ \textbf{if}\ x = 0\ \textbf{then}\ \mathrm{nonzero}()\ \textbf{fi}\ \{IN = L \wedge x = n \wedge n \neq 0\}$$

- From the Conditional rule, this breaks down to
  - $\{R \wedge x = 0\}\ \mathrm{nonzero}()\ \{IN = L \wedge x = n \wedge n \neq 0\}$
  - $(R \wedge x \neq 0) \rightarrow (IN = L \wedge x = n \wedge n \neq 0)$ (obvious)
- The first case involving the recursive call simplifies to

$$\{IN = Z' \cdot n \cdot L \wedge \text{“}Z'\text{ contains only zeros”} \wedge n \neq 0 \wedge x = 0\}$$
$$\mathrm{nonzero}()$$
$$\{IN = L \wedge x = n \wedge n \neq 0\}$$

- The precondition is stronger than we need and $x = 0$ can be removed.

An Example (cont.)

- Finally, we are left with the following proof obligation:

$$\{IN = Z' \cdot n \cdot L \wedge \text{“}Z' \text{ contains only zeros”} \wedge n \neq 0\}$$
$$\mathrm{nonzero}()$$
$$\{IN = L \wedge x = n \wedge n \neq 0\}$$

- The induction hypothesis gives us exactly the above.
- And, this completes the proof.

Termination of Recursive Procedures

- Consider the previous recursive procedure again.

  $\textbf{proc}$ nonzero();
  $\textbf{begin}$
  $\quad\textbf{read}\ x$;
  $\quad\textbf{if}\ x = 0\ \textbf{then}\ \mathrm{nonzero}()\ \textbf{fi}$;
  $\textbf{end}$

- Given an input of the form $IN = L_1 \cdot n \cdot L_2$, where L_1 contains only zero values and $n \neq 0$, the command "nonzero()" will halt.
- We prove this *by induction* on the length of L_1.

Proving Termination by Induction

- Basis: $\mathrm{length}(L_1) = 0$
  - The input has the form $IN = n \cdot L_2$, where $n \neq 0$.
  - After "**read** x", $x \neq 0$.
  - The boolean test $x = 0$ does not pass and the procedure call terminates.
- Induction step: $\mathrm{length}(L_1) = k > 0$
  - Hypothesis: $\mathrm{nonzero}()$ halts when $\mathrm{length}(L_1) = k - 1 \geq 0$.
  - Let $L_1 = 0 \cdot L_1'$.
  - The call $\mathrm{nonzero}()$ is invoked with $IN = 0 \cdot L_1' \cdot n \cdot L_2$, where L_1' contains only zero values and $n \neq 0$.

Proving Termination by Induction (cont.)

- Induction step (cont.)
  - After "$\textbf{read}\ x$", $x = 0$.
  - This boolean test $x = 0$ passes and a second call $\mathrm{nonzero}()$ is invoked inside the $\mathbf{if}$ statement.
  - The second $\mathrm{nonzero}()$ is invoked with $L'_1 \cdot n \cdot L_2$, where L'_1 contains only zero values and $n \neq 0$
  - Since $\mathrm{length}(L'_1) = k - 1$, termination is guaranteed by the hypothesis.

Proving Termination by Induction (cont.)

- A rule for proving termination of recursive procedures:

$$\frac{\{\exists u : W(u < T \wedge P(u))\}\ \mathrm{p}()\ \{Q\} \vdash \{P(T)\}\ S\ \{Q\}}{\vdash \{\exists t : W(P(t))\}\ \mathrm{p}()\ \{Q\}}$$

  where
  - $(W, <)$ is a well-founded set,
  - p is defined as "$\mathbf{proc}\ \mathrm{p}();\ S$", and
  - T is a "rigid" variable that ranges over W and does not occur in P, Q, or S.