Suggested Solutions for Homework Assignment #1

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\land, \lor\}$, $\to$, $\leftrightarrow$, $\vdash$.

1. Prove that every propositional formula has an equivalent formula in the conjunctive normal form and also an equivalent formula in the disjunctive normal form. (Hint: by induction on the structure of a formula, dealing with both cases simultaneously)

   Solution. To be completed. □

2. Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

   (a) $(p \to r) \land (q \to r) \vdash p \lor q \to r$

   Solution.

$$\dfrac{\dfrac{\dfrac{}{(p \to r) \land (q \to r), p \lor q \vdash p \lor q}\,(Hyp) \qquad \alpha \qquad \beta}{(p \to r) \land (q \to r), p \lor q \vdash r}\,(\lor E)}{(p \to r) \land (q \to r) \vdash p \lor q \to r}\,(\to I)$$

α :

$$\dfrac{\dfrac{\dfrac{}{(p \to r) \land (q \to r), p \lor q, p \vdash (p \to r) \land (q \to r)}\,(Hyp)}{(p \to r) \land (q \to r), p \lor q, p \vdash p \to r}\,(\land E_1) \qquad \dfrac{}{(p \to r) \land (q \to r), p \lor q, p \vdash p}\,(Hyp)}{(p \to r) \land (q \to r), p \lor q, p \vdash r}\,(\to E)$$

β :

$$\dfrac{\dfrac{\dfrac{}{(p \to r) \land (q \to r), p \lor q, q \vdash (p \to r) \land (q \to r)}\,(Hyp)}{(p \to r) \land (q \to r), p \lor q, q \vdash q \to r}\,(\land E_2) \qquad \dfrac{}{(p \to r) \land (q \to r), p \lor q, q \vdash q}\,(Hyp)}{(p \to r) \land (q \to r), p \lor q, q \vdash r}\,(\to E)$$

□

   (b) $\vdash (p \land q \to r) \to (p \to (q \to r))$

   Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \land q \to r, p, q \vdash p \land q \to r}\,(Hyp) \qquad \dfrac{\dfrac{}{p \land q \to r, p, q \vdash p}\,(Hyp) \quad \dfrac{}{p \land q \to r, p, q \vdash q}\,(Hyp)}{p \land q \to r, p, q \vdash p \land q}\,(\land I)}{p \land q \to r, p, q \vdash r}\,(\to E)}{p \land q \to r, p \vdash q \to r}\,(\to I)}{p \land q \to r \vdash p \to (q \to r)}\,(\to I)}{\vdash (p \land q \to r) \to (p \to (q \to r))}\,(\to I)$$

□

3. Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

(a) $\vdash (p \to q) \to (\neg p \lor q)$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{}{p \to q, \neg(\neg p \lor q), p \vdash \neg(\neg p \lor q)}\,(Hyp)}{p \to q, \neg(\neg p \lor q), p \vdash (\neg p \lor q) \land \neg(\neg p \lor q)}\,(\land I)}{p \to q, \neg(\neg p \lor q) \vdash \neg p}\,(\neg I)}{p \to q, \neg(\neg p \lor q) \vdash \neg p \lor q}\,(\lor I_1) \qquad \dfrac{}{p \to q, \neg(\neg p \lor q) \vdash \neg(\neg p \lor q)}\,(Hyp)}{p \to q, \neg(\neg p \lor q) \vdash (\neg p \lor q) \land \neg(\neg p \lor q)}\,(\land I)}{\dfrac{p \to q \vdash \neg\neg(\neg p \lor q)}{\dfrac{p \to q \vdash \neg p \lor q}{\vdash (p \to q) \to (\neg p \lor q)}\,(\to I)}\,(\neg\neg E)}\,(\neg I)$$

α :

$$\dfrac{\dfrac{\dfrac{}{p \to q, \neg(\neg p \lor q), p \vdash p \to q}\,(Hyp) \qquad \dfrac{}{p \to q, \neg(\neg p \lor q), p \vdash p}\,(Hyp)}{p \to q, \neg(\neg p \lor q), p \vdash q}\,(\to E)}{p \to q, \neg(\neg p \lor q), p \vdash \neg p \lor q}\,(\lor I_2)$$

□

(b) $\vdash ((p \to q) \to p) \to p$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p \vdash (p \to q) \to p}\,(Hyp) \qquad \alpha}{(p \to q) \to p, \neg p \vdash p}\,(\to E) \qquad \dfrac{}{(p \to q) \to p, \neg p \vdash \neg p}\,(Hyp)}{(p \to q) \to p, \neg p \vdash p \land \neg p}\,(\land I)}{\dfrac{(p \to q) \to p \vdash \neg\neg p}{\dfrac{(p \to q) \to p \vdash p}{\vdash ((p \to q) \to p) \to p}\,(\to I)}\,(\neg\neg E)}\,(\neg I)$$

α :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p, p, \neg q \vdash p}\,(Hyp) \qquad \dfrac{}{(p \to q) \to p, \neg p, p, \neg q \vdash \neg p}\,(Hyp)}{(p \to q) \to p, \neg p, p, \neg q \vdash p \land \neg p}\,(\land I)}{(p \to q) \to p, \neg p, p \vdash \neg\neg q}\,(\neg I)}{\dfrac{(p \to q) \to p, \neg p, p \vdash q}{(p \to q) \to p, \neg p \vdash p \to q}\,(\to I)}\,(\neg\neg E)$$

□