Suggested Solutions for Homework Assignment #2

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\to$, $\leftrightarrow$, $\vdash$.

1. Prove, using *Natural Deduction*, the validity of the following sequents:

   (a) $\forall x(P(x) \to Q(x)) \vdash \forall xP(x) \to \forall xQ(x)$

   Solution. Assume w does not occur free either in $P(x)$ or in $Q(x)$.

$$\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{\dfrac{}{\forall x(P(x) \to Q(x)), \forall xP(x) \vdash \forall xP(x)}\,(Hyp)}{\forall x(P(x) \to Q(x)), \forall xP(x) \vdash P(w)}\,(\forall E)}{\forall x(P(x) \to Q(x)), \forall xP(x) \vdash Q(w)}\,(\to E)}{\forall x(P(x) \to Q(x)), \forall xP(x) \vdash \forall xQ(x)}\,(\forall I)}{\forall x(P(x) \to Q(x)) \vdash \forall xP(x) \to \forall xQ(x)}\,(\to I)$$

   α :

$$\dfrac{\dfrac{}{\forall x(P(x) \to Q(x)), \forall xP(x) \vdash \forall x(P(x) \to Q(x))}\,(Hyp)}{\forall x(P(x) \to Q(x)), \forall xP(x) \vdash P(w) \to Q(w)}\,(\forall E)$$

□

   (b) $\vdash \exists x\forall yP(x, y) \to \forall y\exists xP(x, y)$

   Solution. Assume both w and z do not occur free in $P(x, y)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x\forall yP(x, y) \vdash \exists x\forall yP(x, y)}\,(Hyp) \qquad \dfrac{\dfrac{\dfrac{}{\exists x\forall yP(x, y), \forall yP(z, y) \vdash \forall yP(z, y)}\,(Hyp)}{\exists x\forall yP(x, y), \forall yP(z, y) \vdash P(z, w)}\,(\forall E)}{\exists x\forall yP(x, y), \forall yP(z, y) \vdash \exists xP(x, w)}\,(\exists I)}{\exists x\forall yP(x, y) \vdash \exists xP(x, w)}\,(\exists E)}{\exists x\forall yP(x, y) \vdash \forall y\exists xP(x, y)}\,(\forall I)}{\vdash \exists x\forall yP(x, y) \to \forall y\exists xP(x, y)}\,(\to I)$$

□

   (c) $\forall x(A(x) \to B) \vdash \exists xA(x) \to B$, assuming x does not occur free in B.

   Solution. Assume w does not occur free either in $A(x)$ or in B.

$$\dfrac{\dfrac{\dfrac{}{\forall x(A(x) \to B), \exists xA(x) \vdash \exists xA(x)}\,(Hyp) \qquad \alpha}{\forall x(A(x) \to B), \exists xA(x) \vdash B}\,(\exists E)}{\forall x(A(x) \to B) \vdash \exists xA(x) \to B}\,(\to I)$$

   α :

$$\dfrac{\dfrac{\dfrac{}{\forall x(A(x) \to B), \exists xA(x), A(w) \vdash \forall x(A(x) \to B)}\,(Hyp)}{\forall x(A(x) \to B), \exists xA(x), A(w) \vdash A(w) \to B}\,(\forall E) \qquad \beta}{\forall x(A(x) \to B), \exists xA(x), A(w) \vdash B}\,(\to E)$$

   β :

$$\dfrac{}{\forall x(A(x) \to B), \exists xA(x), A(w) \vdash A(w)}\,(Hyp)$$

$\square$

2. Prove, using *Natural Deduction* for the first-order logic with equality (=), that = is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity), which follows from the =-Introduction rule.

   (a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry)
   Solution.

$$\dfrac{\dfrac{}{t_2 = t_1 \vdash t_2 = t_1}\,(\mathit{Hyp}) \qquad \dfrac{}{t_2 = t_1 \vdash t_2 = t_2}\,(= I)}{t_2 = t_1 \vdash t_1 = t_2}\,(= E)$$

$\square$

   (b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity)
   Solution.

$$\dfrac{\dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_2 = t_3}\,(\mathit{Hyp}) \qquad \dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_2}\,(\mathit{Hyp})}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,(= E)$$

$\square$

3. Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality.

   (a) $\dfrac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,(= \mathit{Symmetry})$
   Solution.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, t_2 = t_1 \vdash t_1 = t_2}\,(\mathit{Axiom}(\mathit{Symmetry}))}{\Gamma \vdash t_2 = t_1 \to t_1 = t_2}\,(\to I) \qquad \Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,(\to E)$$

$\square$

   (b) $\dfrac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,(= \mathit{Transitivity})$
   Solution.

$$\dfrac{\dfrac{\alpha \qquad \Gamma \vdash t_1 = t_2}{\Gamma \vdash t_2 = t_3 \to t_1 = t_3}\,(\to E) \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,(\to E)$$

   α :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,(\mathit{Axiom}(\mathit{Transitivity}))}{\Gamma, t_1 = t_2 \vdash t_2 = t_3 \to t_1 = t_3}\,(\to I)}{\Gamma \vdash t_1 = t_2 \to (t_2 = t_3 \to t_1 = t_3)}\,(\to I)$$

$\square$

4. Prove, using Coq, the validity of the following sequents. You may either print out the Coq proofs or email them to the instructor.

   (a) $\vdash (p \lor q \to r) \to (p \to r) \land (q \to r)$
   (b) $\vdash (p \to (q \to r)) \to (p \land q \to r)$
   (c) $\vdash (\neg p \lor q) \to (p \to q)$

   Solution. To be completed. $\square$