

# Predicate Transformers

(Based on [Dijkstra 1975; Gries 1981; Morgan 1994])

Yih-Kuen Tsay

Department of Information Management  
National Taiwan University

# Basic Idea

- 🌐 The execution of a sequential program, if terminating, **transforms** the **initial** state into some **final** state.
- 🌐 If, for any given postcondition, we know  
*the **weakest precondition** that guarantees termination of the program in a state satisfying the postcondition,*  
then we have fully understood the meaning of the program.

Note: the weakest precondition is **the weakest** in the sense that it identifies **all the desired initial states and nothing else**.



# The Predicate Transformer $wp$

- For a program  $S$  and a predicate (or an assertion)  $Q$ , let  $wp(S, Q)$  denote the aforementioned weakest precondition.
- Therefore, we can see a program as a *predicate transformer*  $wp(S, \cdot)$ , transforming a postcondition  $Q$  (a predicate) into its weakest precondition  $wp(S, Q)$ .
- If the execution of  $S$  starts in a state satisfying  $wp(S, Q)$ , it is guaranteed to terminate and result in a state satisfying  $Q$ .



Note: there is a weaker variant of  $wp$ , called  $wlp$  (weakest liberal precondition), which is defined almost identical to  $wp$  except that termination is not guaranteed.

# Notational Conventions

  $\Rightarrow$  vs.  $\rightarrow$

-   $A \Rightarrow B$  ( $A$  entails  $B$ ) states a relation between two formulae  $A$  and  $B$ : in every state, if  $A$  is true then  $B$  is true.
-   $A \rightarrow B$  is a formula. When “ $A \rightarrow B$ ” stands alone, it usually means  $A \rightarrow B$  is true in every state (model).

  $\equiv$  vs.  $\leftrightarrow$

-   $A \equiv B$  ( $A$  is equivalent to  $B$ ) states a relation between two formulae  $A$  and  $B$ : in every state, if  $A$  is true if and only if  $B$  is true.
-   $A \leftrightarrow B$  is a formula. When “ $A \leftrightarrow B$ ” stands alone, it usually means  $A \leftrightarrow B$  is true in every state (model).

# Hoare Triples in Terms of $wp$

- When total correctness is meant,  $\{P\} S \{Q\}$  can be understood as saying  $P \Rightarrow wp(S, Q)$ .
- In fact, with a suitable formal definition,  $wp$  provides a **semantic** foundation for the Hoare logic.
- The precondition  $P$  here may be as weak as  $wp(S, Q)$ , but often a stronger and easier-to-find  $P$  is all that is needed.

# Properties of $wp$

## Fundamental Properties (Axioms):

 **Law of the Excluded Miracle:**  $wp(S, false) \equiv false$

 **Distributivity of Conjunction:**

$$wp(S, Q_1) \wedge wp(S, Q_2) \equiv wp(S, Q_1 \wedge Q_2)$$

 **Distributivity of Disjunction** for deterministic  $S$ :

$$wp(S, Q_1) \vee wp(S, Q_2) \equiv wp(S, Q_1 \vee Q_2)$$

## Derived Properties:

 **Law of Monotonicity:** if  $Q_1 \Rightarrow Q_2$ , then

$$wp(S, Q_1) \Rightarrow wp(S, Q_2)$$

 **Distributivity of Disjunction** for nondeterministic  $S$ :

$$wp(S, Q_1) \vee wp(S, Q_2) \Rightarrow wp(S, Q_1 \vee Q_2)$$

# Predicate Calculation

🌐 Equivalence is preserved by substituting equals for equals

🌐 Example:

$$\begin{aligned} & (A \vee B) \rightarrow C \\ \equiv & \{ A \rightarrow B \equiv \neg A \vee B \} \\ & \neg(A \vee B) \vee C \\ \equiv & \{ \text{de Morgan's law} \} \\ & (\neg A \wedge \neg B) \vee C \\ \equiv & \{ \text{distributive law} \} \\ & (\neg A \vee C) \wedge (\neg B \vee C) \\ \equiv & \{ A \rightarrow B \equiv \neg A \vee B \} \\ & (A \rightarrow C) \wedge (B \rightarrow C) \end{aligned}$$

# Predicate Calculation (cont.)

Entailment **distributes** over conjunction, disjunction, quantification, and the consequence of an implication.

Example:

$$\begin{aligned}
 & \forall x(A \rightarrow B) \wedge \forall xA \\
 \Rightarrow & \{ \forall x(A \rightarrow B) \Rightarrow (\forall xA \rightarrow \forall xB) \} \\
 & (\forall xA \rightarrow \forall xB) \wedge \forall xA \\
 \equiv & (\neg \forall xA \vee \forall xB) \wedge \forall xA \\
 \equiv & (\neg \forall xA \wedge \forall xA) \vee (\forall xB \wedge \forall xA) \\
 \equiv & \{ \neg A \wedge A \equiv \text{false} \} \\
 & \text{false} \vee (\forall xB \wedge \forall xA) \\
 \equiv & \{ \text{false} \vee A \equiv A \} \\
 & \forall xB \wedge \forall xA \\
 \Rightarrow & \forall xB
 \end{aligned}$$



# Some Laws for Predicate Calculation

🌐 Equivalence is **commutative** and **associative**

$$\odot A \leftrightarrow B \equiv B \leftrightarrow A$$

$$\odot A \leftrightarrow (B \leftrightarrow C) \equiv (A \leftrightarrow B) \leftrightarrow C$$

$$\odot \text{false} \vee A \equiv A \vee \text{false} \equiv A$$

$$\odot \neg A \wedge A \equiv \text{false}$$

$$\odot A \rightarrow B \equiv \neg A \vee B$$

$$\odot A \rightarrow \text{false} \equiv \neg A$$

$$\odot (A \vee B) \rightarrow C \equiv (A \rightarrow C) \wedge (B \rightarrow C)$$

$$\odot A \rightarrow (B \rightarrow C) \equiv (A \wedge B) \rightarrow C$$

$$\odot A \rightarrow B \equiv A \leftrightarrow (A \wedge B)$$

$$\odot A \wedge B \Rightarrow A$$

🌐  $\forall x(x = E \rightarrow A) \equiv A[E/x] \equiv \exists x(x = E \wedge A)$ , if  $x$  is not free in  $E$ .

🌐  $\forall x(A \wedge B) \equiv \forall xA \wedge \forall xB$

🌐  $\forall x(A \rightarrow B) \Rightarrow \forall xA \rightarrow \forall xB$

🌐  $\forall x(A \rightarrow B) \equiv A \rightarrow \forall xB$ , if  $x$  is not free in  $A$ .

🌐  $\exists x(A \wedge B) \equiv A \wedge \exists xB$ , if  $x$  is not free in  $A$ .

# “Extreme” Programs

$$\text{wp}(\mathbf{skip}, Q) \triangleq Q$$

$$\text{wp}(\mathbf{choose } x, x \in \text{Dom}(x)) \triangleq \mathit{true}$$

$$\text{wp}(\mathbf{choose } x, Q) \triangleq Q, \text{ if } x \text{ is not free in } Q$$

$$\text{wp}(\mathbf{abort}, Q) \triangleq \mathit{false}$$

# The Assignment Statement

🌐 Syntax:  $x := E$

Note: this becomes a multiple assignment, if we view  $x$  as a list of distinct variables and  $E$  as a list of expressions.


🌐 Semantics:  $wp(x := E, Q) \triangleq Q[E/x]$ .

🌐 Syntax:  $S_1; S_2$

🌐 Semantics:  $wp(S_1; S_2, Q) \triangleq wp(S_1, wp(S_2, Q))$ .


# Abbreviation of Conjunctions/Disjunctions

## Conjunction:


 Original Form:  $B_1 \wedge B_2 \wedge \cdots \wedge B_n$

 Abbreviation:  $\forall i : 1 \leq i \leq n : B_i$

## Disjunction:

 Original Form:  $B_1 \vee B_2 \vee \cdots \vee B_n$

 Abbreviation:  $\exists i : 1 \leq i \leq n : B_i$

 This applies to conjunctions/disjunctions of first-order formulae, Hoare triples, etc.

# The Alternative Statement

🌐 Syntax:

$$\begin{array}{l} \text{IF: } \mathbf{if} \ B_1 \rightarrow S_1 \\ \quad \quad \quad \square B_2 \rightarrow S_2 \\ \quad \quad \quad \dots \\ \quad \quad \quad \square B_n \rightarrow S_n \\ \quad \quad \quad \mathbf{fi} \end{array}$$

Each of the “ $B_i \rightarrow S_i$ ”s is a guarded command, where  $B_i$  is the guard (a boolean expression) and  $S_i$  the command (body).

🌐 Informal description: One of the guarded commands, whose guard evaluates to true, is **nondeterministically selected** and the corresponding command **executed**. If none of the guards evaluates to true, then the execution **aborts**.

# The Alternative Statement (cont.)

🌐 Syntax:

$$\begin{array}{l} \text{IF: } \mathbf{if} \ B_1 \rightarrow S_1 \\ \quad \quad \square B_2 \rightarrow S_2 \\ \quad \quad \dots \\ \quad \quad \square B_n \rightarrow S_n \\ \quad \mathbf{fi} \end{array}$$

🌐 Semantics:

$$\begin{aligned} wp(\text{IF}, Q) &\triangleq (\exists i : 1 \leq i \leq n : B_i) \\ &\quad \wedge (\forall i : 1 \leq i \leq n : B_i \rightarrow wp(S_i, Q)) \end{aligned}$$

🌐 The case of simple IF:

$$wp(\mathbf{if} \ B \rightarrow S \ \mathbf{fi}, Q) \triangleq B \wedge (B \rightarrow wp(S, Q))$$



# The Alternative Statement (cont.)

Suppose there exists a predicate  $P$  such that

1.  $P \Rightarrow (\exists i : 1 \leq i \leq n : B_i)$  and
2.  $\forall i : 1 \leq i \leq n : P \wedge B_i \Rightarrow wp(S_i, Q)$ .

Then  $P \Rightarrow wp(IF, Q)$ .

The less obvious part is  $P \Rightarrow (\forall i : 1 \leq i \leq n : B_i \rightarrow wp(S_i, Q))$ .

$$\begin{aligned} & \forall i : 1 \leq i \leq n : (P \wedge B_i) \rightarrow wp(S_i, Q) \\ \equiv & \forall i : 1 \leq i \leq n : P \rightarrow (B_i \rightarrow wp(S_i, Q)) \\ \equiv & P \rightarrow (\forall i : 1 \leq i \leq n : B_i \rightarrow wp(S_i, Q)) \end{aligned}$$

# The Alternative Statement (cont.)

- 🌐 Inference rule in the Hoare logic:

$$\frac{P \Rightarrow (\exists i : 1 \leq i \leq n : B_i) \quad \forall i : 1 \leq i \leq n : \{P \wedge B_i\} S_i \{Q\}}{\{P\} \text{ IF : } \mathbf{if} B_1 \rightarrow S_1 [] \cdots [] B_n \rightarrow S_n \mathbf{fi} \{Q\}}$$

- 🌐 This rule follows from the preceding theorem.
- 🌐 The case of simple IF:

$$\frac{P \Rightarrow B \quad \{P \wedge B\} S \{Q\}}{\{P\} \mathbf{if} B \rightarrow S \mathbf{fi} \{Q\}}$$

# The Iterative Statement

🌐 Syntax:

DO: **do**  $B_1 \rightarrow S_1$   
     $\square$   $B_2 \rightarrow S_2$   
    ...  
     $\square$   $B_n \rightarrow S_n$   
**od**

Each of the “ $B_i \rightarrow S_i$ ”s is a guarded command.

- 🌐 Informal description: Choose (nondeterministically) a guard  $B_i$  that evaluates to true and execute the corresponding command  $S_i$ . If none of the guards evaluates to true, then the execution **terminates**.
- 🌐 The usual “**while**  $B$  **do**  $S$  **od**” can be defined as this simple *while*-loop: “**do**  $B \rightarrow S$  **od**”.

# The Iterative Statement (cont.)

Let  $BB$  denote  $\exists i : 1 \leq i \leq n : B_i$ , i.e.,  $B_1 \vee B_2 \vee \dots \vee B_n$ .

The  $DO$  statement is equivalent to

```
do  $BB \rightarrow$  if  $B_1 \rightarrow S_1$   
       $[] B_2 \rightarrow S_2$   
      ...  
       $[] B_n \rightarrow S_n$   
if
```

**od**

or simply **do**  $BB \rightarrow IF$  **od**.

This suggests that we could have got by with just the simple *while*-loop.

# The Iterative Statement (cont.)

- Again, let BB denote  $\exists i : 1 \leq i \leq n : B_i$ .
- Let  $H_k(Q)$ ,  $k \geq 0$ , be defined as follows.

$$\begin{cases} H_0(Q) \triangleq \neg \text{BB} \wedge Q \\ H_k(Q) \triangleq H_0(Q) \vee wp(\text{IF}, H_{k-1}(Q)) \quad \text{for } k > 0 \end{cases}$$

- The predicate  $H_0(Q)$  represents the set of states where execution of DO terminates immediately (0 iteration).
- The predicate  $H_k(Q)$ , for  $k > 0$ , represents the set of states where execution of DO terminates after at most  $k$  iterations.
- Semantics of DO:

$$wp(\text{DO}, Q) \triangleq (\exists k : 0 \leq k : H_k(Q))$$

# A More Useful Theorem for DO

Suppose there exist a predicate  $P$  and an integer-valued expression  $t$  such that

1.  $\forall i : 1 \leq i \leq n : P \wedge B_i \Rightarrow wp(S_i, P)$ ,
2.  $P \Rightarrow (t \geq 0)$ , and
3.  $\forall i : 1 \leq i \leq n : P \wedge B_i \wedge (t = t_0) \Rightarrow wp(S_i, t < t_0)$ , where  $t_0$  is a rigid variable.

Then  $P \Rightarrow wp(\text{DO}, P \wedge \neg \text{BB})$ .

$$\begin{aligned}
 P &\equiv P \wedge (\exists k : 0 \leq k : t \leq k) && (t \text{ is finite}) \\
 &\equiv \exists k : 0 \leq k : P \wedge t \leq k && (k \text{ is not free in } P) \\
 &\Rightarrow \exists k : 0 \leq k : H_k(P \wedge \neg \text{BB}) && (P \wedge t \leq k \Rightarrow H_k(P \wedge \neg \text{BB})) \\
 &\equiv wp(\text{DO}, P \wedge \neg \text{BB}) && (\text{def. of DO})
 \end{aligned}$$

# A More Useful Theorem for DO (cont.)

- 🌐 Proof of  $P \wedge t \leq k \Rightarrow H_k(P \wedge \neg \text{BB})$  is by induction on  $k$ .
- 🌐 Will do this for the case of simple DO.

# A Simplified Theorem for Simple DO

Suppose there exist a predicate  $P$  and an integer-valued expression  $t$  such that

1.  $P \wedge B \Rightarrow wp(S, P)$ ,
2.  $P \Rightarrow (t \geq 0)$ , and
3.  $P \wedge B \wedge (t = t_0) \Rightarrow wp(S, t < t_0)$ , where  $t_0$  is a rigid variable.

Then  $P \Rightarrow wp(\mathbf{do} B \rightarrow S \mathbf{od}, P \wedge \neg B)$ .

This is to be contrasted by

$$\frac{\{P \wedge B\} S \{P\} \quad \{P \wedge B \wedge t = Z\} S \{t < Z\} \quad P \Rightarrow (t \geq 0)}{\{P\} \mathbf{while} B \mathbf{do} S \mathbf{od} \{P \wedge \neg B\}}$$



# A Simplified Theorem for Simple DO (cont.)



Proof of  $P \wedge t \leq k \Rightarrow H_k(P \wedge \neg B)$  is by induction on  $k$ .

Recall, for simple DO,

$$\begin{cases} H_0(Q) \triangleq \neg B \wedge Q \\ H_k(Q) \triangleq H_0(Q) \vee wp(\mathbf{if} B \rightarrow S \mathbf{fi}, H_{k-1}(Q)) \quad \text{for } k > 0 \end{cases}$$

# A Simplified Theorem for Simple DO (cont.)

- Base case:  $P \wedge t \leq 0 \Rightarrow H_0(P \wedge \neg B)$ , which is equivalent to  $P \wedge t \leq 0 \Rightarrow P \wedge \neg B$ .

Since  $P \Rightarrow (t \geq 0)$ , it suffices to show that  $P \wedge t = 0 \Rightarrow \neg B$ .

$$\begin{aligned} & P \wedge t = 0 \wedge B \\ \equiv & (P \wedge B) \wedge (P \wedge B \wedge t = 0) \\ \Rightarrow & wp(S, P) \wedge wp(S, t < 0) \\ \equiv & wp(S, P \wedge t < 0) \\ \equiv & wp(S, false) \\ \equiv & false \end{aligned}$$

# A Simplified Theorem for Simple DO (cont.)

- 🌍 Inductive step ( $k > 0$ ):  $P \wedge t \leq k \Rightarrow H_k(P \wedge \neg B)$ , i.e.,  
 $P \wedge t \leq k \Rightarrow H_0(P \wedge \neg B) \vee wp(\mathbf{if} B \rightarrow S \mathbf{fi}, H_{k-1}(P \wedge \neg B))$ .

Split  $P \wedge t \leq k$  into three cases:



- ☀️  $P \wedge (t \leq k - 1)$
- ☀️  $P \wedge B \wedge (t = k)$ 
  - $\Rightarrow B \wedge (B \rightarrow wp(S, P)) \wedge B \wedge (B \rightarrow wp(S, t < k))$
  - $\Rightarrow wp(\mathbf{if} B \rightarrow S \mathbf{fi}, P) \wedge wp(\mathbf{if} B \rightarrow S \mathbf{fi}, t < k)$
  - $\equiv wp(\mathbf{if} B \rightarrow S \mathbf{fi}, P \wedge t < k)$
  - $\equiv wp(\mathbf{if} B \rightarrow S \mathbf{fi}, P \wedge (t \leq k - 1))$
  - $\Rightarrow \{ \text{Ind. Hypothesis and Monotonicity of } wp \}$
  - $wp(\mathbf{if} B \rightarrow S \mathbf{fi}, H_{k-1}(P \wedge \neg B))$
  - $\Rightarrow H_0(P \wedge \neg B) \vee wp(\mathbf{if} B \rightarrow S \mathbf{fi}, H_{k-1}(P \wedge \neg B))$
- ☀️  $P \wedge \neg B \wedge (t = k)$

# Refinement

## Syntax:

$$prog_1 \sqsubseteq prog_2$$

which is read as “ $prog_1$  *is refined by*  $prog_2$ ” or “ $prog_2$  *refines*  $prog_1$ ” ( $prog_2 \sqsupseteq prog_1$ ).


-  Informal description: intuitively, the refinement relation conveys the concept of program  $prog_2$  being better than  $prog_1$ . Program  $prog_2$  is better in the sense that it is more accurate, applies in more situations, or runs more efficiently.
-  A program may be derived through a series of refinement steps.

# Specifications


## Syntax:


$$w : [pre, post]$$


where *pre* is the precondition, *post* is postcondition, and the “*w*” part is called the *frame*.

 Informal description: the specification describes an *abstract* program such that if the initial state satisfies the precondition *pre*, then it *changes only variables listed in the frame* and terminates in a final state satisfying the postcondition *post*.

## Examples:

  $y : [0 \leq x \leq 9, y^2 = x]$

  $y : [0 \leq x, y^2 = x \wedge y \geq 0]$

  $x : [true, x = x_0 + 1 \vee x = x_0 - 1]$  ( $x_0$  denotes the initial value of  $x$ )

# Some Laws for Refinement

 strengthen postcondition: If  $post' \Rightarrow post$ , then

$$w : [pre, post] \sqsubseteq w : [pre, post']$$

Example:


$$y : [0 \leq x \leq 9, y^2 = x] \sqsubseteq y : [0 \leq x \leq 9, y^2 = x \wedge y \geq 0]$$

 weaken precondition: If  $pre \Rightarrow pre'$ , then

$$w : [pre, post] \sqsubseteq w : [pre', post]$$


Example:

$$y : [0 \leq x \leq 9, y^2 = x \wedge y \geq 0] \sqsubseteq y : [0 \leq x, y^2 = x \wedge y \geq 0]$$

 Combining the two refinements,


$$y : [0 \leq x \leq 9, y^2 = x] \sqsubseteq y : [0 \leq x, y^2 = x \wedge y \geq 0]$$

# Some Laws for Refinement (cont.)

 assignment: If  $pre \Rightarrow post[E/x]$ , then

$$w, x : [pre, post] \sqsubseteq x := E$$

Note:  $w$  may (but not necessarily) be changed.

 sequential composition: For any predicate  $mid$ ,

$$w : [pre, post] \sqsubseteq w : [pre, mid]; w : [mid, post]$$

# Semantics of Specification

🌐 Syntax:  $w : [pre, post]$

🌐 Semantics:

$$wp(w : [pre, post], Q) \triangleq pre \wedge (\forall w(post \rightarrow Q))[v/v_0]$$

where the substitution  $[v/v_0]$  replaces all “initial” variables, i.e.,  $v_0$ , by corresponding final variables.

Note: initial variables  $v_0$  do not occur in  $Q$ .

🌐 Example:  $wp(x := x \pm 1, Q) \equiv Q[x + 1/x] \wedge Q[x - 1/x]$



# Semantics of Specification (cont.)

$$\begin{aligned}
 & wp(x := x \pm 1, Q) \\
 \equiv & wp(x : [true, x = x_0 + 1 \vee x = x_0 - 1], Q) \\
 \equiv & \{ \text{def. of specification} \} \\
 & true \wedge \forall x((x = x_0 + 1 \vee x = x_0 - 1) \rightarrow Q)[x/x_0] \\
 \equiv & \forall x((x = x_0 + 1 \rightarrow Q) \wedge (x = x_0 - 1 \rightarrow Q))[x/x_0] \\
 \equiv & (\forall x(x = x_0 + 1 \rightarrow Q) \wedge \forall x(x = x_0 - 1 \rightarrow Q))[x/x_0] \\
 \equiv & \forall x(x = x_0 + 1 \rightarrow Q)[x/x_0] \wedge \forall x(x = x_0 - 1 \rightarrow Q)[x/x_0] \\
 \equiv & \{ \forall x(x = E \rightarrow A) \equiv A[E/x] \} \\
 & (Q[x_0 + 1/x])[x/x_0] \wedge (Q[x_0 - 1/x])[x/x_0] \\
 \equiv & \{ Q \text{ does not contain } x_0 \} \\
 & Q[x + 1/x] \wedge Q[x - 1/x]
 \end{aligned}$$

# Semantics of Refinement

🌐 Syntax:  $prog_1 \sqsubseteq prog_2$

🌐 Semantics: for all  $Q$ ,

$$wp(prog_1, Q) \Rightarrow wp(prog_2, Q)$$

🌐 Examples:

☀️  $x := x \pm 1 \sqsubseteq x := x + 1$

$$\begin{aligned} & wp(x := x \pm 1, Q) \\ \equiv & Q[x + 1/x] \wedge Q[x - 1/x] \\ \Rightarrow & Q[x + 1/x] \\ \equiv & wp(x := x + 1, Q) \end{aligned}$$

☀️  $x := x \pm 1 \sqsubseteq x := x - 1$