

## Suggested Solutions for Homework Assignment #1

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order:  $\neg$ ,  $\{\wedge, \vee\}$ ,  $\rightarrow$ ,  $\leftrightarrow$ ,  $\vdash$ .

- (30 points) Prove that every propositional formula has an equivalent formula in the conjunctive normal form and also an equivalent formula in the disjunctive normal form. (Hint: by induction on the structure of a formula, dealing with both cases simultaneously)

*Solution.* Let us first review some preliminaries. A (non-empty) clause is a disjunction of one or more literals such as  $p \vee \neg q \vee r$ , while a (non-empty) term/product is a conjunction of one or more literals such as  $\neg p \wedge q \wedge \neg r$ . (Note: the name “term” as defined here is not commonly used in propositional logic. However, it is adequate in light of the notion of a term in algebraic expressions. An alternative name is “product”.) So, a formula is in conjunctive normal form (CNF) if it is a conjunction of one or more clauses. A formula is in disjunctive normal form (DNF) if it is a disjunction of one or more terms. A clause by itself is in CNF (a one-clause CNF) and, when seen as a disjunction of one-literal terms, is also in DNF. Similarly, for a term. A single literal is a special case of a clause and also of a term.

The complement of a clause (term), after the negation is pushed to the literal level, becomes a term (clause), e.g.,  $\neg(p \vee \neg q \vee r) \Leftrightarrow \neg p \wedge q \wedge \neg r$ . Taking this one level up, the complement of a formula in CNF (DNF), after the negation is pushed to the literal level, becomes a formula in DNF (CNF), e.g.,  $\neg((p \vee \neg q) \wedge (q \vee r)) \Leftrightarrow (\neg p \wedge q) \vee (\neg q \wedge \neg r)$ .

Now we prove the problem statement by induction on the structure of a given formula  $\varphi$ .

Base case ( $\varphi$  is just a propositional symbol): a propositional symbol can be seen as a single-literal clause or term and so is already in CNF and in DNF.

Inductive step: there are three cases.

- $\varphi = \neg\psi$ : let  $\psi^C$  be a formula equivalent to  $\psi$  in CNF and  $\psi^D$  an equivalent formula in DNF (guaranteed to exist by the induction hypothesis). Pushing the negation at the front of  $\neg\psi^C$  ( $\neg\psi^D$ ) to the literal level, we get a formula equivalent to  $\varphi$  in DNF (CNF).
- $\varphi = \varphi_1 \wedge \varphi_2$ : let  $\varphi_1^C$  ( $\varphi_2^C$ ) be a formula equivalent to  $\varphi_1$  ( $\varphi_2$ ) in CNF and  $\varphi_1^D$  ( $\varphi_2^D$ ) an equivalent formula in DNF. The formula  $\varphi_1^C \wedge \varphi_2^C$  is equivalent to  $\varphi$  and readily in CNF.

To obtain a formula equivalent to  $\varphi$  in DNF, suppose  $\varphi_1^D = t_1 \vee t_2 \vee \dots \vee t_l$  and  $\varphi_2^D = u_1 \vee u_2 \vee \dots \vee u_m$ , where  $t_i$ 's and  $u_j$ 's are terms. Then, by repeatedly distributing the top-level  $\wedge$  in  $\varphi_1^D \wedge \varphi_2^D$  to the term level, we obtain a formula  $\bigvee_{1 \leq i \leq l, 1 \leq j \leq m} (t_i \wedge u_j)$  in DNF that is equivalent to  $\varphi$ .

- $\varphi = \varphi_1 \vee \varphi_2$ : analogous to the case of  $\varphi = \varphi_1 \wedge \varphi_2$ .

□

- (40 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

(a)  $p \vee q \rightarrow r \vdash (p \rightarrow r) \wedge (q \rightarrow r)$

*Solution.*

$$\frac{\frac{\alpha}{p \vee q \rightarrow r \vdash p \rightarrow r} (\rightarrow I) \quad \frac{\beta}{p \vee q \rightarrow r \vdash q \rightarrow r} (\rightarrow I)}{p \vee q \rightarrow r \vdash (p \rightarrow r) \wedge (q \rightarrow r)} (\wedge I)$$

$\alpha$  :

$$\frac{\frac{}{p \vee q \rightarrow r, p \vdash p \vee q \rightarrow r} (Hyp) \quad \frac{}{p \vee q \rightarrow r, p \vdash p} (Hyp)}{\frac{}{p \vee q \rightarrow r, p \vdash p \vee q} (\vee I_1)}{p \vee q \rightarrow r, p \vdash r} (\rightarrow E)}$$

$\beta$  :

$$\frac{\frac{}{p \vee q \rightarrow r, q \vdash p \vee q \rightarrow r} (Hyp) \quad \frac{}{p \vee q \rightarrow r, q \vdash q} (Hyp)}{\frac{}{p \vee q \rightarrow r, q \vdash p \vee q} (\vee I_2)}{p \vee q \rightarrow r, q \vdash r} (\rightarrow E)}$$

□

(b)  $\vdash (p \wedge q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$

*Solution.*

$$\frac{\frac{}{p \wedge q \rightarrow r, p, q \vdash p \wedge q \rightarrow r} (Hyp) \quad \frac{\frac{}{p \wedge q \rightarrow r, p, q \vdash p} (Hyp) \quad \frac{}{p \wedge q \rightarrow r, p, q \vdash q} (Hyp)}{p \wedge q \rightarrow r, p, q \vdash p \wedge q} (\wedge I)}{\frac{}{p \wedge q \rightarrow r, p, q \vdash p \wedge q} (\rightarrow E)}{\frac{}{p \wedge q \rightarrow r, p, q \vdash r} (\rightarrow I)}{\frac{}{p \wedge q \rightarrow r, p \vdash q \rightarrow r} (\rightarrow I)}{\frac{}{p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)} (\rightarrow I)}{\vdash (p \wedge q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))} (\rightarrow I)}$$

□

3. (30 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

(a)  $\vdash (\neg p \vee q) \rightarrow (p \rightarrow q)$

*Solution.*

$$\frac{\frac{}{\neg p \vee q, p \vdash \neg p \vee q} (Hyp) \quad \alpha \quad \frac{}{\neg p \vee q, p, q \vdash q} (Hyp)}{\frac{}{\neg p \vee q, p \vdash q} (\vee E)}{\frac{}{\neg p \vee q, p \vdash p \rightarrow q} (\rightarrow I)}{\vdash (\neg p \vee q) \rightarrow (p \rightarrow q)} (\rightarrow I)}$$

$\alpha$  :

$$\frac{\frac{}{\neg p \vee q, p, \neg p \vdash p} (Hyp) \quad \frac{}{\neg p \vee q, p, \neg p \vdash \neg p} (Hyp)}{\neg p \vee q, p, \neg p \vdash q} (\rightarrow E)$$

□

(b)  $\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$ *Solution.*

$$\begin{array}{c}
\frac{}{(p \rightarrow q) \rightarrow p, \neg p \vdash (p \rightarrow q) \rightarrow p} \text{ (Hyp)} \quad \alpha \quad \frac{}{(p \rightarrow q) \rightarrow p, \neg p \vdash \neg p} \text{ (Hyp)} \\
\frac{}{(p \rightarrow q) \rightarrow p, \neg p \vdash p} \text{ (}\rightarrow E\text{)} \quad \frac{}{(p \rightarrow q) \rightarrow p, \neg p \vdash \neg p} \text{ (}\wedge I\text{)} \\
\frac{}{(p \rightarrow q) \rightarrow p, \neg p \vdash p \wedge \neg p} \text{ (}\neg I\text{)} \\
\frac{}{(p \rightarrow q) \rightarrow p \vdash \neg \neg p} \text{ (}\neg \neg E\text{)} \\
\frac{}{(p \rightarrow q) \rightarrow p \vdash p} \text{ (}\neg I\text{)} \\
\frac{}{\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p} \text{ (}\rightarrow I\text{)}
\end{array}$$

 $\alpha :$ 

$$\begin{array}{c}
\frac{}{(p \rightarrow q) \rightarrow p, \neg p, p, \neg q \vdash p} \text{ (Hyp)} \quad \frac{}{(p \rightarrow q) \rightarrow p, \neg p, p, \neg q \vdash \neg p} \text{ (Hyp)} \\
\frac{}{(p \rightarrow q) \rightarrow p, \neg p, p \vdash q} \text{ (}\rightarrow I\text{)} \\
\frac{}{(p \rightarrow q) \rightarrow p, \neg p \vdash p \rightarrow q} \text{ (}\neg E\text{)}
\end{array}$$

□