Homework Assignment #2

Due Time/Date

2:20PM Wednesday, October 7, 2020. Late submission will be penalized by 20% for each working day overdue.

Note

Please write or type your answers on A4 (or similar size) paper. Put your completed homework on the instructor's desk before the class starts. For late submissions, please drop them in Yih-Kuen Tsay's mail box on the first floor of Management Building 2. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. (30 points) Prove, using *Natural Deduction*, the validity of the following sequents:
   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall x P(x) \rightarrow \forall x Q(x)$
   (b) $\vdash \exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$
   (c) $\forall x(A(x) \rightarrow B) \vdash \exists x A(x) \rightarrow B$, assuming x does not occur free in B.
2. (20 points) Prove, using *Natural Deduction* for the first-order logic with equality (=), that = is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity), which follows from the =-Introduction rule.
   (a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry)
   (b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity)
3. (20 points) Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality.
   (a) $\dfrac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}$ (= *Symmetry*)
   (b) $\dfrac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}$ (= *Transitivity*)

4. (30 points) A first-order theory for *groups* contains the following three axioms:

- $\forall a \forall b \forall c (a \cdot (b \cdot c) = (a \cdot b) \cdot c)$. (Associativity)
- $\forall a ((a \cdot e = a) \wedge (e \cdot a = a))$. (Identity)
- $\forall a ((a \cdot a^{-1} = e) \wedge (a^{-1} \cdot a = e))$. (Inverse)

Here $\cdot$ is the binary operation, e is a constant, called the identity, and $(\cdot)^{-1}$ is the inverse function which gives the inverse of an element. Let M denote the set of the three axioms. Prove, using *Natural Deduction* plus the derived rules in the preceding problem, the validity of the following sequents:

(a) $M \vdash \forall a \forall b \forall c ((a \cdot b = a \cdot c) \to b = c)$. (Hint: a typical proof in algebra books is the following: $b = e \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c) = (a^{-1} \cdot a) \cdot c = e \cdot c = c$.)

(b) $M \vdash \forall a \forall b \forall c (((a \cdot b = e) \wedge (b \cdot a = e) \wedge (a \cdot c = e) \wedge (c \cdot a = e)) \to b = c)$, which says that every element has a unique inverse. (Hint: a typical proof in algebra books is the following: $b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c$.)