Suggested Solutions for Homework Assignment #2

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\to$, $\leftrightarrow$, $\vdash$.

1. (30 points) Prove, using *Natural Deduction*, the validity of the following sequents:

   (a) $\forall x(P(x) \to Q(x)) \vdash \forall x P(x) \to \forall x Q(x)$

   Solution. Assume w does not occur free either in $P(x)$ or in $Q(x)$.

$$\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{\dfrac{}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x P(x)}\,(Hyp)}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash P(w)}\,(\forall E)}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash Q(w)}\,(\to E)}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)}\,(\forall I)}{\forall x(P(x) \to Q(x)) \vdash \forall x P(x) \to \forall x Q(x)}\,(\to I)$$

   α :

$$\dfrac{\dfrac{}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x(P(x) \to Q(x))}\,(Hyp)}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash P(w) \to Q(w)}\,(\forall E)$$

□

   (b) $\vdash \exists x \forall y P(x, y) \to \forall y \exists x P(x, y)$

   Solution. Assume both w and z do not occur free in $P(x, y)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x \forall y P(x, y) \vdash \exists x \forall y P(x, y)}\,(Hyp) \qquad \dfrac{\dfrac{\dfrac{}{\exists x \forall y P(x, y), \forall y P(z, y) \vdash \forall y P(z, y)}\,(Hyp)}{\exists x \forall y P(x, y), \forall y P(z, y) \vdash P(z, w)}\,(\forall E)}{\exists x \forall y P(x, y), \forall y P(z, y) \vdash \exists x P(x, w)}\,(\exists I)}{\exists x \forall y P(x, y) \vdash \exists x P(x, w)}\,(\exists E)}{\exists x \forall y P(x, y) \vdash \forall y \exists x P(x, y)}\,(\forall I)}{\vdash \exists x \forall y P(x, y) \to \forall y \exists x P(x, y)}\,(\to I)$$

□

   (c) $\forall x(A(x) \to B) \vdash \exists x A(x) \to B$, assuming x does not occur free in B.

   Solution. Assume w does not occur free either in $A(x)$ or in B.

$$\dfrac{\dfrac{\dfrac{}{\forall x(A(x) \to B), \exists x A(x) \vdash \exists x A(x)}\,(Hyp) \qquad \alpha}{\forall x(A(x) \to B), \exists x A(x) \vdash B}\,(\exists E)}{\forall x(A(x) \to B) \vdash \exists x A(x) \to B}\,(\to I)$$

   α :

$$\dfrac{\dfrac{\dfrac{}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash \forall x(A(x) \to B)}\,(Hyp)}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash A(w) \to B}\,(\forall E) \qquad \beta}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash B}\,(\to E)$$

   β :

$$\dfrac{}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash A(w)}\,(Hyp)$$

$\square$

2. Prove, using *Natural Deduction* for the first-order logic with equality (=), that = is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity), which follows from the =-Introduction rule.

   (a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry)
   Solution.

$$\dfrac{\dfrac{}{t_2 = t_1 \vdash t_2 = t_1}\,(Hyp) \qquad \dfrac{}{t_2 = t_1 \vdash t_2 = t_2}\,(=I)}{t_2 = t_1 \vdash t_1 = t_2}\,(=E)$$

$\square$

   (b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity)
   Solution.

$$\dfrac{\dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_2 = t_3}\,(Hyp) \qquad \dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_2}\,(Hyp)}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,(=E)$$

$\square$

3. (20 points) Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality.

   (a) $\dfrac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,(=\mathit{Symmetry})$
   Solution.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, t_2 = t_1 \vdash t_1 = t_2}\,(Axiom(Symmetry))}{\Gamma \vdash t_2 = t_1 \rightarrow t_1 = t_2}\,(\rightarrow I) \qquad \Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,(\rightarrow E)$$

$\square$

   (b) $\dfrac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,(=\mathit{Transitivity})$
   Solution.

$$\dfrac{\dfrac{\alpha \qquad \Gamma \vdash t_1 = t_2}{\Gamma \vdash t_2 = t_3 \rightarrow t_1 = t_3}\,(\rightarrow E) \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,(\rightarrow E)$$

   α :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,(Axiom(Transitivity))}{\Gamma, t_1 = t_2 \vdash t_2 = t_3 \rightarrow t_1 = t_3}\,(\rightarrow I)}{\Gamma \vdash t_1 = t_2 \rightarrow (t_2 = t_3 \rightarrow t_1 = t_3)}\,(\rightarrow I)$$

$\square$

4. (30 points) A first-order theory for *groups* contains the following three axioms:

   - $\forall a \forall b \forall c (a \cdot (b \cdot c) = (a \cdot b) \cdot c)$. (Associativity)
   - $\forall a ((a \cdot e = a) \wedge (e \cdot a = a))$. (Identity)
   - $\forall a ((a \cdot a^{-1} = e) \wedge (a^{-1} \cdot a = e))$. (Inverse)

Here $\cdot$ is the binary operation, e is a constant, called the identity, and $(\cdot)^{-1}$ is the inverse function which gives the inverse of an element. Let M denote the set of the three axioms. Prove, using *Natural Deduction* plus the derived rules in the preceding problem, the validity of the following sequents:

(a) $M \vdash \forall a \forall b \forall c((a \cdot b = a \cdot c) \to b = c)$. (Hint: a typical proof in algebra books is the following: $b = e \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c) = (a^{-1} \cdot a) \cdot c = e \cdot c = c$.)

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \qquad \delta}{M, x \cdot y = x \cdot z \vdash y = z}\,(=E)}{M \vdash (x \cdot y = x \cdot z) \to y = z}\,(\to I)}{M \vdash \forall c((x \cdot y = x \cdot c) \to y = c)}\,(\forall I)}{M \vdash \forall b \forall c((x \cdot b = x \cdot c) \to b = c)}\,(\forall I)}{M \vdash \forall a \forall b \forall c((a \cdot b = a \cdot c) \to b = c)}\,(\forall I)$$

α :

$$\dfrac{\dfrac{\beta \qquad \gamma}{M, x \cdot y = x \cdot z \vdash (x^{-1} \cdot x) \cdot y = y}\,(=E) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash \forall a \forall b \forall c (a \cdot (b \cdot c) = (a \cdot b) \cdot c)}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash \forall b \forall c (x^{-1} \cdot (b \cdot c) = (x^{-1} \cdot b) \cdot c)}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash \forall c (x^{-1} \cdot (x \cdot c) = (x^{-1} \cdot x) \cdot c)}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) \cdot y}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot y) = y}\,(=E)$$

β :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash \forall a (a \cdot a^{-1} = e \wedge a^{-1} \cdot a = e)}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot x = e}\,(\wedge E_2)}{M, x \cdot y = x \cdot z \vdash e = x^{-1} \cdot x}\,(= Symmetry)$$

γ :

$$\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash \forall a (a \cdot e = a \wedge e \cdot a = a)}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash y \cdot e = y \wedge e \cdot y = y}\,(\forall E)}{M, x \cdot y = x \cdot z \vdash e \cdot y = y}\,(\wedge E_2)$$

δ :

$$\dfrac{\dfrac{\dfrac{}{M, x \cdot y = x \cdot z \vdash x \cdot y = x \cdot z}\,(Hyp)}{M, x \cdot y = x \cdot z \vdash x \cdot z = x \cdot y}\,(= Symmetry) \qquad \dfrac{\text{the proof tree is similar to } \alpha}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot z) = z}}{M, x \cdot y = x \cdot z \vdash x^{-1} \cdot (x \cdot y) = z}\,(=E)$$

□

(b) $M \vdash \forall a \forall b \forall c(((a \cdot b = e) \wedge (b \cdot a = e) \wedge (a \cdot c = e) \wedge (c \cdot a = e)) \to b = c)$, which says that every element has a unique inverse. (Hint: a typical proof in algebra books is the following: $b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c$.)

Solution. We use N to denote $x \cdot y = e \wedge y \cdot x = e \wedge x \cdot z = e \wedge z \cdot x = e$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)\alpha \qquad (1)\delta}{M, N, x \cdot y = x \cdot z \vdash y = z}\;(=E)}{M, N \vdash x \cdot y = x \cdot z \to y = z}\;(\to I) \qquad \dfrac{\alpha \qquad \beta}{M, N \vdash x \cdot y = x \cdot z}\;(=E)}{M, N \vdash y = z}\;(\to E)}{M \vdash (x \cdot y = e \wedge y \cdot x = e \wedge x \cdot z = e \wedge z \cdot x = e) \to y = z}\;(\to I)}{M \vdash \forall c((x \cdot y = e \wedge y \cdot x = e \wedge x \cdot c = e \wedge c \cdot x = e) \to y = c)}\;(\forall I)}{M \vdash \forall b \forall c((x \cdot b = e \wedge b \cdot x = e \wedge x \cdot c = e \wedge c \cdot x = e) \to b = c)}\;(\forall I)}{M \vdash \forall a \forall b \forall c((a \cdot b = e \wedge b \cdot a = e \wedge a \cdot c = e \wedge c \cdot a = e) \to b = c)}\;(\forall I)}{}
$$

α :

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{M, N \vdash x \cdot y = e \wedge y \cdot x = e \wedge x \cdot z = e \wedge z \cdot x = e}\;(Hyp)}{M, N \vdash x \cdot z = e \wedge z \cdot x = e}\;(\wedge E_2)}{M, N \vdash x \cdot z = e}\;(\wedge E_1)}{M, N \vdash e = x \cdot z}\;(= Symmetry)
$$

β :

$$
\dfrac{\dfrac{}{M, N \vdash x \cdot y = e \wedge y \cdot x = e \wedge x \cdot z = e \wedge z \cdot x = e}\;(Hyp)}{M, N \vdash x \cdot y = e}\;(\wedge E_1)
$$

□