Suggested Solutions for Homework Assignment #1

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. (30 points) Prove that every propositional formula has an equivalent formula in the conjunctive normal form and also an equivalent formula in the disjunctive normal form. (Hint: by induction on the structure of a formula, dealing with both cases simultaneously)

   Solution. Let us first review some preliminaries. A (non-empty) clause is a disjunction of one or more literals such as $p \vee \neg q \vee r$, while a (non-empty) term/product is a conjunction of one or more literals such as $\neg p \wedge q \wedge \neg r$. (Note: the name "term" as defined here is not commonly used in propositional logic. However, it is adequate in light of the notion of a term in algebraic expressions. An alternative name is "product".) So, a formula is in conjunctive normal form (CNF) if it is a conjunction of one or more clauses. A formula is in disjunctive normal form (DNF) if it is a disjunction of one or more terms. A clause by itself is in CNF (a one-clause CNF) and, when seen as a disjunction of one-literal terms, is also in DNF. Similarly, for a term. A single literal is a special case of a clause and also of a term.

   The complement of a clause (term), after the negation is pushed to the literal level, becomes a term (clause), e.g., $\neg(p \vee \neg q \vee r) \Leftrightarrow \neg p \wedge q \wedge \neg r$. Taking this one level up, the complement of a formula in CNF (DNF), after the negation is pushed to the literal level, becomes a formula in DNF (CNF), e.g., $\neg((p \vee \neg q) \wedge (q \vee r)) \Leftrightarrow (\neg p \wedge q) \vee (\neg q \wedge \neg r)$.

   Now we prove the problem statement by induction on the structure of a given formula φ.

   Base case (φ is just a propositional symbol): a propositional symbol can be seen as a single-literal clause or term and so is already in CNF and in DNF.

   Inductive step: there are three cases.

   (a) $\varphi = \neg\psi$: let ψ^C be a formula equivalent to ψ in CNF and ψ^D an equivalent formula in DNF (guaranteed to exist by the induction hypothesis). Pushing the negation at the front of $\neg\psi^C$ ($\neg\psi^D$) to the literal level, we get a formula equivalent to φ in DNF (CNF).

   (b) $\varphi = \varphi_1 \wedge \varphi_2$: let φ_1^C (φ_2^C) be a formula equivalent to φ_1 (φ_2) in CNF and φ_1^D (φ_2^D) an equivalent formula in DNF. The formula $\varphi_1^C \wedge \varphi_2^C$ is equivalent to φ and readily in CNF.

   To obtain a formula equivalent to φ in DNF, suppose $\varphi_1^D = t_1 \vee t_2 \vee \cdots \vee t_l$ and $\varphi_2^D = u_1 \vee u_2 \vee \cdots \vee u_m$, where t_i's and u_j's are terms. Then, by repeatedly distributing the top-level $\wedge$ in $\varphi_1^D \wedge \varphi_2^D$ to the term level, we obtain a formula $\bigvee_{1 \leq i \leq l, 1 \leq j \leq m}(t_i \wedge u_j)$ in DNF that is equivalent to φ.

   (c) $\varphi = \varphi_1 \vee \varphi_2$: analogous to the case of $\varphi = \varphi_1 \wedge \varphi_2$.

   □

2. (40 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

(a) $p \lor q \to r \vdash (p \to r) \land (q \to r)$

Solution.

$$\dfrac{\dfrac{\alpha}{p \lor q \to r \vdash p \to r}\,(\to I) \qquad \dfrac{\beta}{p \lor q \to r \vdash q \to r}\,(\to I)}{p \lor q \to r \vdash (p \to r) \land (q \to r)}\,(\land I)$$

α :

$$\dfrac{\dfrac{}{p \lor q \to r, p \vdash p \lor q \to r}\,(Hyp) \qquad \dfrac{\dfrac{}{p \lor q \to r, p \vdash p}\,(Hyp)}{p \lor q \to r, p \vdash p \lor q}\,(\lor I_1)}{p \lor q \to r, p \vdash r}\,(\to E)$$

β :

$$\dfrac{\dfrac{}{p \lor q \to r, q \vdash p \lor q \to r}\,(Hyp) \qquad \dfrac{\dfrac{}{p \lor q \to r, q \vdash q}\,(Hyp)}{p \lor q \to r, q \vdash p \lor q}\,(\lor I_2)}{p \lor q \to r, q \vdash r}\,(\to E)$$

□

(b) $\vdash (p \to (q \to r)) \to (p \land q \to r)$

Solution.

$$\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{\dfrac{}{p \to (q \to r), p \land q \vdash p \land q}\,(Hyp)}{p \to (q \to r), p \land q \vdash q}\,(\land E_2)}{p \to (q \to r), p \land q \vdash r}\,(\to E)}{p \to (q \to r) \vdash p \land q \to r}\,(\to I)}{\vdash (p \to (q \to r)) \to (p \land q \to r)}\,(\to I)$$

α :

$$\dfrac{\dfrac{}{p \to (q \to r), p \land q \vdash p \to (q \to r)}\,(Hyp) \qquad \dfrac{\dfrac{}{p \to (q \to r), p \land q \vdash p \land q}\,(Hyp)}{p \to (q \to r), p \land q \vdash p}\,(\land E_1)}{p \to (q \to r), p \land q \vdash q \to r}\,(\to E)$$

□

3. (30 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

(a) $\vdash (p \to q) \to (\neg p \lor q)$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{}{p \to q, \neg(\neg p \vee q), p \vdash \neg(\neg p \vee q)}\,(Hyp)}{p \to q, \neg(\neg p \vee q), p \vdash (\neg p \vee q) \wedge \neg(\neg p \vee q)}\,(\wedge I)}{p \to q, \neg(\neg p \vee q) \vdash \neg p}\,(\neg I)}{p \to q, \neg(\neg p \vee q) \vdash \neg p \vee q}\,(\vee I_1) \qquad \dfrac{}{p \to q, \neg(\neg p \vee q) \vdash \neg(\neg p \vee q)}\,(Hyp)}{\dfrac{\dfrac{\dfrac{p \to q, \neg(\neg p \vee q) \vdash (\neg p \vee q) \wedge \neg(\neg p \vee q)}{p \to q \vdash \neg\neg(\neg p \vee q)}\,(\neg I)}{p \to q \vdash \neg p \vee q}\,(\neg\neg E)}{\vdash (p \to q) \to (\neg p \vee q)}\,(\to I)}\,(\wedge I)$$

α :

$$\dfrac{\dfrac{\dfrac{}{p \to q, \neg(\neg p \vee q), p \vdash p \to q}\,(Hyp) \qquad \dfrac{}{p \to q, \neg(\neg p \vee q), p \vdash p}\,(Hyp)}{p \to q, \neg(\neg p \vee q), p \vdash q}\,(\to E)}{p \to q, \neg(\neg p \vee q), p \vdash \neg p \vee q}\,(\vee I_2)$$

□

(b) $\vdash ((p \to q) \to p) \to p$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p \vdash (p \to q) \to p}\,(Hyp) \qquad \alpha}{(p \to q) \to p, \neg p \vdash p}\,(\to E) \qquad \dfrac{}{(p \to q) \to p, \neg p \vdash \neg p}\,(Hyp)}{(p \to q) \to p, \neg p \vdash p \wedge \neg p}\,(\wedge I)}{(p \to q) \to p \vdash \neg\neg p}\,(\neg I)}{\dfrac{(p \to q) \to p \vdash p}{\vdash ((p \to q) \to p) \to p}\,(\to I)}\,(\neg\neg E)$$

α :

$$\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p, p, \neg q \vdash p}\,(Hyp) \qquad \dfrac{}{(p \to q) \to p, \neg p, p, \neg q \vdash \neg p}\,(Hyp)}{(p \to q) \to p, \neg p, p \vdash q}\,(\neg E)}{(p \to q) \to p, \neg p \vdash p \to q}\,(\to I)$$

□