Homework Assignment #2

Due Time/Date

2:20PM Wednesday, September 28, 2022. Late submission will be penalized by 20% for each working day overdue.

How to Submit

Please use a word processor or scan hand-written answers to produce a single PDF file. Name your file according to this pattern: "b097050xx-hw2". Upload the PDF file to the NTU COOL site for Software Specification and Verification 2022. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. (20 points) Please provide a precise description, using formulae in first-order logic, for each of the following requirements. The functions/constants and predicates you may use are: $+$, $\times$, 0, 1, 2, $<$, $=$, $\leq$, plus those introduced in the requirement statements. Make assumptions where you see necessary.

   (a) The array $A[0..N-1]$ (of integers) represents a max heap with $A[0]$ as the root.

   (b) The array $A[0..N-1]$ (of integers) is cyclically sorted in an increasing order. (Note: $3, 4, 0, 1, 2$, for example, is a cyclically sorted list of integers.)

2. (20 points) Prove, using *Natural Deduction*, the validity of the following sequents:

   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall x P(x) \rightarrow \forall x Q(x)$

   (b) $\vdash \exists x \forall y P(x,y) \rightarrow \forall y \exists x P(x,y)$

3. (20 points) Prove, using *Natural Deduction* for the first-order logic with equality ($=$), that $=$ is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity), which follows from the $=$-Introduction rule.

   (a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry)

   (b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity)

4. (20 points) Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality.

(a) $$\frac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2} \ (= \textit{Symmetry})$$

(b) $$\frac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3} \ (= \textit{Transitivity})$$

5. (20 points) A first-order theory for *groups* contains the following three axioms:

- $\forall a \forall b \forall c (a \cdot (b \cdot c) = (a \cdot b) \cdot c)$. (Associativity)
- $\forall a ((a \cdot e = a) \wedge (e \cdot a = a))$. (Identity)
- $\forall a ((a \cdot a^{-1} = e) \wedge (a^{-1} \cdot a = e))$. (Inverse)

Here $\cdot$ is the binary operation, e is a constant, called the identity, and $(\cdot)^{-1}$ is the inverse function which gives the inverse of an element. Let M denote the set of the three axioms. Prove, using *Natural Deduction* plus the derived rules in the preceding problem, the validity of the following sequent:

$M \vdash \forall a \forall b \forall c ((a \cdot b = a \cdot c) \to b = c)$.

(Hint: a typical proof in algebra books is the following: $b = e \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c) = (a^{-1} \cdot a) \cdot c = e \cdot c = c$.)