Temporal Verification of Reactive Systems

(Based on Manna and Pnueli [1991,1995,1996])

Yih-Kuen Tsay

Department of Information Management
National Taiwan University

Computational vs. Reactive Programs

- Computational (Transformational) Programs
  - Run to produce a final result on termination
  - An example:

$$
\begin{array}{l}
[\ \mathrm{local}\ x : \mathrm{integer\ initially}\ x = n; \\
\quad y := 0; \\
\quad \textbf{while}\ x > 0\ \textbf{do} \\
\qquad\qquad x, y := x - 1, y + 2x - 1 \\
\quad \textbf{od}\ ]
\end{array}
$$

  - Only the initial values and the (final) result are relevant to correctness
  - Can be specified by pre and post-conditions such as
    - $\{n \geq 0\}\ y := ?\ \{y = n^2\}$ or
    - $y : [n \geq 0, y = n^2]$

Computational vs. Reactive Programs (cont.)

- Reactive Programs
  - Maintaining an ongoing (typically not terminating) interaction with their environments
  - An example:

$$s : \{0,1\}\ \textbf{initially}\ s = 1$$

$$\left[\begin{array}{l} l_0 : \textbf{loop forever do} \\ \quad \left[\begin{array}{ll} l_1 : & \text{remainder;} \\ l_2 : & \text{request}(s); \\ l_3 : & \text{critical;} \\ l_4 : & \text{release}(s); \end{array}\right] \end{array}\right] \parallel \left[\begin{array}{l} m_0 : \textbf{loop forever do} \\ \quad \left[\begin{array}{ll} m_1 : & \text{remainder;} \\ m_2 : & \text{request}(s); \\ m_3 : & \text{critical;} \\ m_4 : & \text{release}(s); \end{array}\right] \end{array}\right]$$

  - Must be specified and verified in terms of their behaviors, including the intermediate states

The Framework

- Computational Model: for providing an abstract syntactic base
  - fair transition systems (FTS)
  - fair discrete systems (FDS)
- Implementation Language: for describing the actual implementation; will define syntax by examples; translated into FTS or FDS for verification
- Specification Language: for specifying properties of a system; will use linear temporal logic (LTL)
- Verification Techniques: for verifying that an implementation satisfies its specification
  - algorithmic methods: state space exploration
  - deductive methods: mathematical theorem proving

Three Kinds of Validity

IM NTU

- Assertional Validity: validity of non-temporal formulae, i.e., state formulae, over an arbitrary state (valuation)
- General Temporal Validity: validity of temporal formulae over arbitrary sequences of states
- Program Validity: validity of a temporal formula over sequence of states that represent computations of the analyzed system

Variables

- Three kinds of variables will be needed:
  - Program (system) variables
  - Primed version of program variables: for referring to the values of program variables in the next state when defining a state transition
  - Specification variables: appearing only in formulae (but not in the program) that specify properties of a program
- We assume that all these variables are drawn from a universal set of variables $\mathcal{V}$.
- For every unprimed variable $x \in \mathcal{V}$, its primed version x' is also in $\mathcal{V}$.
- Each variable has a type.

Assertions

- For describing a system and its specification, we assume an underlying first-order assertion language over $\mathcal{V}$.
- The language provides the following elements:
  - Expressions (corresponding to first-order terms):
    variables, constants, and functions applied to expressions
  - Atomic formulae:
    propositions or boolean variables and predicates applied to expressions
  - Assertions or state formulae (corresponding to first-order formulae):
    atomic formulae, boolean connectives applied to formulae, and quantifiers applied to formulae

Fair Transition Systems

A fair transition system (FTS) $\mathcal{P}$ is a tuple $\langle V, \Theta, \mathcal{T}, \mathcal{J}, \mathcal{C} \rangle$:

- $V \subseteq \mathcal{V}$: a finite set of typed state variables, including *data* and *control* variables. A (type-respecting) valuation of V is called a *V-state* or simply *state*. The set of all V-states is denoted Σ_V.
- Θ : the initial condition, an assertion characterizing the *initial states*.
- $\mathcal{T}$: a set of transitions, including the *idling* transition. Each transition is associated with a *transition relation*, relating a state and its successor state(s).
- $\mathcal{J} \subseteq \mathcal{T}$: a set of just (weakly fair) transitions.
- $\mathcal{C} \subseteq \mathcal{T}$: a set of compassionate (strongly fair) transitions.

Transitions of an FTS

The transition relation of a transition $\tau \in \mathcal{T}$ is expressed as an assertion $\rho_\tau(V, V')$:

- Example: $x = 1 \wedge x' = 0$.
  For $s, s' \in \Sigma_V$, $\langle s, s' \rangle \models x = 1 \wedge x' = 0$ holds if the value of x is 1 in state s and the value of x is 0 in (the next) state s'.
- τ-successor
  - State s' is a *τ-successor* of s if $\langle s, s' \rangle \models \rho_\tau(V, V')$
  - $\tau(s) \triangleq \{s' \mid s' \text{ is a } \tau\text{-successor of } s\}$.
- enabledness of τ
  - $En(\tau) \triangleq (\exists V')\rho_\tau(V, V')$.
  - τ is enabled in a state if $En(\tau)$ holds in that state.
  - τ is enabled in state s iff s has some τ-successor.

Computations of an FTS

Given an FTS $\mathcal{P} = \langle V, \Theta, \mathcal{T}, \mathcal{J}, \mathcal{C} \rangle$, a computation of $\mathcal{P}$ is an infinite sequence of states $\sigma : s_0, s_1, s_2, \cdots$ satisfying:

- Initiation: s_0 is an initial state, i.e., $s_0 \models \Theta$.
- Consecution: for every $i \geq 0$, s_{i+1} is a τ-successor of state s_i, i.e., $\langle s_i, s_{i+1} \rangle \models \rho_\tau(V, V')$, for some $\tau \in \mathcal{T}$. In this case, we say that τ is *taken* at position i.
- Justice: for every $\tau \in \mathcal{J}$, it is never the case that τ is continuously enabled, but never taken, from some point on.
- Compassion: for every $\tau \in \mathcal{C}$, it is never the case that τ is enabled infinitely often, but never taken, from some point on.

The set of all computations of $\mathcal{P}$ is denoted by $Comp(\mathcal{P})$.

An Example Program and Its FTS

- Program ANY-Y:

$$x, y : \text{natural } \textbf{initially } x = y = 0$$

$$\begin{bmatrix} l_0 : \textbf{while } x = 0 \textbf{ do} \\ \left[\ l_1 : \quad y := y + 1;\ \right] \\ l_2 : \end{bmatrix} \parallel \begin{bmatrix} m_0 : x := 1 \\ m_2 : \end{bmatrix}$$

- Informal description:
  - The program consists of an *asynchronous composition* of two processes.
  - One process continuously increments y as long as it finds x to be 0, while the other simply sets x to 1 (when it gets its turn to execute).
  - The executions of the program are all possible *interleavings* of the steps of the individual processes.

An Example Program and Its FTS (cont.)

- Program ANY-Y as an FTS $\mathcal{P}_{\text{ANY-Y}} = \langle V, \Theta, \mathcal{T}, \mathcal{J}, \mathcal{C} \rangle$:
  - $V \triangleq \{x, y : \text{natural}, \pi_0 : \{l_0, l_1, l_2\}, \pi_1 : \{m_0, m_1\}\}$
  - $\Theta \triangleq \pi_0 = l_0 \wedge \pi_1 = m_0 \wedge x = y = 0$
  - $\mathcal{T} \triangleq \{\tau_I, \tau_{l_0}, \tau_{l_1}, \tau_{m_0}\}$, whose transition relations are

$$\left.\begin{array}{ll} \rho_I : & \pi'_0 = \pi_0 \wedge \pi'_1 = \pi_1 \wedge x' = x \wedge y' = y \;, \\ \rho_{l_0} : & \pi_0 = l_0 \wedge ((x = 0 \wedge \pi'_0 = l_1) \vee (x \neq 0 \wedge \pi'_0 = l_2)) \\ & \wedge \pi'_1 = \pi_1 \wedge x' = x \wedge y' = y \end{array}\right. , \text{ etc.}$$

  - $\mathcal{J} \triangleq \{\tau_{l_0}, \tau_{l_1}, \tau_{m_0}\}$
  - $\mathcal{C} \triangleq \emptyset$

Program Mux

$$Q_0, Q_1 : \text{bool } \textbf{initially } Q_0 = Q_1 = \mathit{false}$$
$$T : \{0,1\} \textbf{ initially } T = 0$$

$$P_0 :: \left[\begin{array}{l} l_0 : \textbf{loop forever do} \\ \quad \left[\begin{array}{ll} l_1 : & \text{remainder;} \\ l_2 : & Q_0 := \mathit{true}; \\ l_3 : & T := 0; \\ l_4 : & \textbf{await } \neg Q_1 \lor T \neq 0; \\ l_5 : & \text{critical;} \\ l_6 : & Q_0 := \mathit{false}; \end{array}\right] \end{array}\right] \quad \| \quad P_1 :: \left[\begin{array}{l} m_0 : \textbf{loop forever do} \\ \quad \left[\begin{array}{ll} m_1 : & \text{remainder;} \\ m_2 : & Q_1 := \mathit{true}; \\ m_3 : & T := 1; \\ m_4 : & \textbf{await } \neg Q_0 \lor T \neq 1; \\ m_5 : & \text{critical;} \\ m_6 : & Q_1 := \mathit{false}; \end{array}\right] \end{array}\right]$$

Justice is sufficient in preventing individual starvation.

Strong Fairness (Compassion) Is Needed

- Program $\textsc{Mux-Sem}$: mutual exclusion by a semaphore.

$$s : \text{natural } \textbf{initially } s = 1$$

$$\left[\begin{array}{l} l_0 : \textbf{loop forever do} \\ \left[\begin{array}{ll} l_1 : & \text{remainder}; \\ l_2 : & \text{request}(s); \\ l_3 : & \text{critical}; \\ l_4 : & \text{release}(s); \end{array}\right] \end{array}\right] \parallel \left[\begin{array}{l} m_0 : \textbf{loop forever do} \\ \left[\begin{array}{ll} m_1 : & \text{remainder}; \\ m_2 : & \text{request}(s); \\ m_3 : & \text{critical}; \\ m_4 : & \text{release}(s); \end{array}\right] \end{array}\right]$$

  - $\text{request}(s) \triangleq \langle \textbf{await } s > 0 : s := s - 1 \rangle$
  - $\text{release}(s) \triangleq s := s + 1$
- $\mathcal{C}$: $\{\tau_{l_2}, \tau_{m_2}\}$

Linear Temporal Logic (LTL)

- State formulae
  Constructed from the underlying assertion language
- Temporal formulae
  - All state formulae are also temporal formulae.
  - If p and q are temporal formulae and x a variable in $\mathcal{V}$, then the following are temporal formulae:
    - $\neg p$, $p \vee q$, $p \wedge q$, $p \rightarrow q$, $p \leftrightarrow q$
    - $\bigcirc p$, $\Diamond p$, $\Box p$, $p \,\mathcal{U}\, q$, $p \,\mathcal{W}\, q$
    - $\circleddash p$, $\tilde{\circleddash} p$, $\diamondminus p$, $\boxminus p$, $p \,\mathcal{S}\, q$, $p \,\mathcal{B}\, q$
    - $\exists x : p$, $\forall x : p$

Semantics of LTL

IM NTU

- Temporal formulae are interpreted over an infinite sequence of states, called a model, with respect to a position in that sequence.
- We will define the satisfaction relation $(\sigma, i) \models \varphi$ (or φ holds in (σ, i)), as the formal semantics of a temporal formula φ over an infinite sequence of states $\sigma = s_0, s_1, s_2, \ldots, s_i, \ldots$ and a position $i \geq 0$.
- A sequence σ *satisfies* a temporal formula φ, denoted $\sigma \models \varphi$, if $(\sigma, 0) \models \varphi$.
- Variables in $\mathcal{V}$ are partitioned into *flexible* and *rigid* variables. A flexible variable may assume different values in different states, while a rigid variable must assume the same value in all states of a model.

Semantics of LTL (cont.)

- For a state formula p:
  $(\sigma, i) \models p \iff p$ holds at s_i.
- Boolean combinations of formulae:
  $(\sigma, i) \models \neg p \iff (\sigma, i) \models p$ does not hold.
  $(\sigma, i) \models p \vee q \iff (\sigma, i) \models p$ or $(\sigma, i) \models q$.
  $(\sigma, i) \models p \wedge q \iff (\sigma, i) \models p$ and $(\sigma, i) \models q$.
  $(\sigma, i) \models p \rightarrow q \iff (\sigma, i) \models p$ implies $(\sigma, i) \models q$.
  $(\sigma, i) \models p \leftrightarrow q \iff (\sigma, i) \models p$ if and only if $(\sigma, i) \models q$.

  Alternatively, the latter three cases can be defined in terms of $\neg$ and $\vee$, namely $p \wedge q \triangleq \neg(\neg p \vee \neg q)$, $p \rightarrow q \triangleq \neg p \vee q$, and $p \leftrightarrow q \triangleq (p \rightarrow q) \wedge (q \rightarrow p)$.

Semantics of LTL: Future Operators

- $\bigcirc p$ (next p):
  $(\sigma, i) \models \bigcirc p \iff (\sigma, i+1) \models p$.
- $\Diamond p$ (eventually p or sometime p):
  $(\sigma, i) \models \Diamond p \iff$ for some $k \geq i$, $(\sigma, k) \models p$.
- $\Box p$ (henceforth p or always p):
  $(\sigma, i) \models \Box p \iff$ for every $k \geq i$, $(\sigma, k) \models p$.
- $p \, \mathcal{U} \, q$ (p until q):
  $(\sigma, i) \models p \, \mathcal{U} \, q \iff$ for some $k \geq i$, $(\sigma, k) \models q$ and for every j s.t. $i \leq j < k$, $(\sigma, j) \models p$.
- $p \, \mathcal{W} \, q$ (p wait-for q):
  $(\sigma, i) \models p \, \mathcal{W} \, q \iff$ for every $k \geq i$, $(\sigma, k) \models p$, or for some $k \geq i$, $(\sigma, k) \models q$ and for every j, $i \leq j < k$, $(\sigma, j) \models p$.

Semantics of LTL: Future Operators (cont.)

- It can be shown that, for every σ and i,
  - $(\sigma, i) \models \Diamond p$ iff $(\sigma, i) \models \mathit{true}\ \mathcal{U}\ p$
  - $(\sigma, i) \models \Box p$ iff $(\sigma, i) \models \neg\Diamond\neg p$
  - $(\sigma, i) \models p\ \mathcal{W}\ q$ iff $(\sigma, i) \models \Box p \vee p\ \mathcal{U}\ q$
- So, one can also take $\bigcirc$ and $\mathcal{U}$ as the primitive operators and define others in terms of $\bigcirc$ and $\mathcal{U}$:
  - $\Diamond p \triangleq \mathit{true}\ \mathcal{U}\ p$
  - $\Box p \triangleq \neg\Diamond\neg p$
  - $p\ \mathcal{W}\ q \triangleq \Box p \vee p\ \mathcal{U}\ q$

Semantics of LTL: Past Operators

- $\ominus p$ (previous p):
  $(\sigma, i) \models \ominus p \iff (i > 0)$ and $(\sigma, i-1) \models p$.
- $\widetilde{\ominus} p$ (before p):
  $(\sigma, i) \models \widetilde{\ominus} p \iff (i > 0)$ implies $(\sigma, i-1) \models p$.
- $\diamondminus p$ (once p):
  $(\sigma, i) \models \diamondminus p \iff$ for some k, $0 \leq k \leq i$, $(\sigma, k) \models p$.
- $\boxminus p$ (so-far p):
  $(\sigma, i) \models \boxminus p \iff$ for every k, $0 \leq k \leq i$, $(\sigma, k) \models p$.
- $p \, \mathcal{S} \, q$ (p since q):
  $(\sigma, i) \models p \, \mathcal{S} \, q \iff$ for some k, $0 \leq k \leq i$, $(\sigma, k) \models q$ and for every j, $k < j \leq i$, $(\sigma, j) \models p$.

Semantics of LTL: Past Operators (cont.)

- $p \; \mathcal{B} \; q$ (p back-to q):
  $(\sigma, i) \models p \; \mathcal{B} \; q \iff$ for every k, $0 \leq k \leq i$, $(\sigma, k) \models p$, or for some k, $0 \leq k \leq i$, $(\sigma, k) \models q$ and for every j, $k < j \leq i$, $(\sigma, j) \models p$.

Semantics of LTL: Past Operators (cont.)

- It can be shown that, for every σ and i,
  - $(\sigma, i) \models \ominus p$ iff $(\sigma, i) \models \neg \tilde{\ominus} \neg p$
  - $(\sigma, i) \models \diamondminus p$ iff $(\sigma, i) \models \mathit{true}\ \mathcal{S}\ p$
  - $(\sigma, i) \models \boxminus p$ iff $(\sigma, i) \models \neg \diamondminus \neg p$
  - $(\sigma, i) \models p\ \mathcal{B}\ q$ iff $(\sigma, i) \models \boxminus p \vee p\ \mathcal{S}\ q$
- So, one can also take $\tilde{\ominus}$ and $\mathcal{S}$ as the primitive operators and define others in terms of $\tilde{\ominus}$ and $\mathcal{S}$:
  - $\ominus p \triangleq \neg \tilde{\ominus} \neg p$
  - $\diamondminus p \triangleq \mathit{true}\ \mathcal{S}\ p$
  - $\boxminus p \triangleq \neg \diamondminus \neg p$
  - $p\ \mathcal{B}\ q \triangleq \boxminus p \vee p\ \mathcal{S}\ q$

Semantics of LTL: Quantifiers

A sequence σ' is called a *u-variant* of σ if σ' differs from σ in at most the interpretation given to u in each state.

- $(\sigma, i) \models \exists u\colon \varphi \iff (\sigma', i) \models \varphi$ for some *u-variant* σ' of σ.
- $(\sigma, i) \models \forall u\colon \varphi \iff (\sigma', i) \models \varphi$ for every *u-variant* σ' of σ.

  Alternatively, $\forall u\colon \varphi \triangleq \neg(\exists u\colon \neg\varphi)$.

These definitions apply to both flexible and rigid variables.

Some LTL Conventions

- Let *first* abbreviate $\circleddash$*false*, which holds only at position 0; *first* means "this is the first state".
- We use u^- to denote the previous value of u; by convention, u^- equals u at position 0.
  - Example: $x = x^- + 1$.
  - In pure LTL,
    $(\mathit{first} \wedge x = x + 1) \vee (\neg \mathit{first} \wedge \forall u\colon \ominus(x = u) \to x = u + 1)$.
- We use u^+ (or u') to denote the next value of u, i.e., the value of u at the next position.
  - Example: $x^+ = x + 1$.
  - In pure LTL, $\forall u\colon x = u \to \bigcirc(x = u + 1)$.
- These previous and next-value notations also apply to expressions.

Validity

- A state formula is *state valid* if it holds in every state.
- A temporal formula p is (temporally) *valid*, denoted $\models p$, if it holds in every model.
- A state formula is P-*state valid* if it holds in every P-accessible state (i.e., every state that appears in some computation of P).
- A temporal formula p is P-*valid*, denoted $P \models p$, if it holds in every computation of P.

Equivalence and Congruence

- Two formulae p and q are *equivalent* if $p \leftrightarrow q$ is valid.
  Example: $p \mathcal{W} q \leftrightarrow \Box(\diamondminus \neg p \rightarrow \diamondminus q)$.
- Two formulae p and q are *congruent* if $\Box(p \leftrightarrow q)$ is valid.
  Example: $\neg\Diamond p$ and $\Box\neg p$ are congruent, as $\Box(\neg\Diamond p \leftrightarrow \Box\neg p)$ is valid.
- Two congruent formulae may replace each other in any context.

A Hierarchy of Temporal Properties

- Classes of temporal properties; p, q, p_i, q_i below are arbitrary past temporal formulae
  - Safety properties: $\Box p$
  - Guarantee properties: $\Diamond p$
  - Obligation properties: $\bigwedge_{i=1}^{n}(\Box p_i \lor \Diamond q_i)$
  - Response properties: $\Box\Diamond p$
  - Persistence properties: $\Diamond\Box p$
  - Reactivity properties: $\bigwedge_{i=1}^{n}(\Box\Diamond p_i \lor \Diamond\Box q_i)$
- The hierarchy

$$\begin{array}{c} \text{Safety} \\ \text{Guarantee} \end{array} \subseteq \text{Obligation} \subseteq \begin{array}{c} \text{Response} \\ \text{Persistence} \end{array} \subseteq \text{Reactivity}$$

- Every temporal formula is equivalent to some reactivity formula.

More Common Temporal Properties

- Safety properties: $\Box p$
  Example: $p \ \mathcal{W} \ q$ is a safety property, as it is equivalent to $\Box(\diamondminus\neg p \rightarrow \diamondminus q)$.
- Response properties
  - Canonical form: $\Box\Diamond p$
  - Variant: $\Box(p \rightarrow \Diamond q)$ (p leads-to q), which is equivalent to $\Box\Diamond(\neg p \ \mathcal{B} \ q)$.
- Reactivity properties: $\bigwedge_{i=1}^{n}(\Box\Diamond p_i \vee \Diamond\Box q_i)$
- (Simple) reactivity properties
  - Canonical form: $\Box\Diamond p \vee \Diamond\Box q$
  - Variants: $\Box\Diamond p \rightarrow \Box\Diamond q$ or $\Box(\Box\Diamond p \rightarrow \Diamond q)$, which is equivalent to $\Box\Diamond q \vee \Diamond\Box\neg p$.
  - Extended form: $\Box((p \wedge \Box\Diamond r) \rightarrow \Diamond q)$

Rules for Safety Properties

Rule INV

$$\begin{array}{ll} \text{I1.} & \Theta \rightarrow \varphi \\ \text{I2.} & \varphi \rightarrow q \\ \text{I3.} & \{\varphi\}\ \mathcal{T}\ \{\varphi\} \\ \hline & \Box q \end{array}$$

where $\{p\}\ \mathcal{T}\ \{q\}$ means $\{p\}\ \tau\ \{q\}$ (i.e., $\rho_\tau \wedge p \rightarrow q'$) for every $\tau \in \mathcal{T}$

- The auxiliary assertion φ is called an *inductive invariant*, as it holds initially and is preserved by every transition.
- This rule is sound and (relatively) complete for establishing P-validity of the future safety formula $\Box q$ (where q is a state formula).

A Safety Property of Program Mux-Sem

- Mutual exclusion: $\Box(\neg(\pi_0 = l_3 \wedge \pi_1 = m_3))$, which is not inductive.
- The inductive φ needed:

$$y \geq 0 \wedge (\pi_0 = l_3) + (\pi_0 = l_4) + (\pi_1 = m_3) + (\pi_1 = m_4) + y = 1$$

where *true* and *false* are equated respectively with 1 and 0.

Rules for Response Properties

Rule J-RESP (for a just transition $\tau \in \mathcal{J}$)

$$
\begin{array}{ll}
\text{J1.} & \Box(p \rightarrow (q \vee \varphi)) \\
\text{J2.} & \{\varphi\}\ \mathcal{T}\ \{q \vee \varphi\} \\
\text{J3.} & \{\varphi\}\ \tau\ \{q\} \\
\text{J4.} & \Box(\varphi \rightarrow (q \vee En(\tau))) \\
\hline
 & \Box(p \rightarrow \Diamond q)
\end{array}
$$

This is a "one-step" rule that relies on a helpful just transition.

Rules for Response Properties (cont.)

Analogously, there is a one-step rule that relies on a helpful compassionate transition.

Rule C-RESP (for a compassionate transition $\tau \in \mathcal{C}$)

$$
\begin{array}{ll}
\mathrm{C1.} & \Box(p \rightarrow (q \vee \varphi)) \\
\mathrm{C2.} & \{\varphi\}\ \mathcal{T}\ \{q \vee \varphi\} \\
\mathrm{C3.} & \{\varphi\}\ \tau\ \{q\} \\
\mathrm{C4.} & \mathcal{T} - \{\tau\} \vdash \Box(\varphi \rightarrow \Diamond(q \vee En(\tau))) \\
\hline
 & \Box(p \rightarrow \Diamond q)
\end{array}
$$

Premise C4 states that the proof obligation should be carried out for a smaller program with $\mathcal{T} - \{\tau\}$ as the set of transitions.

Rules for Response Properties (cont.)

Rule M-RESP (monotonicity) and Rule T-RESP (transitivity)

$$\frac{\begin{array}{l}\Box(p \to r), \Box(t \to q) \\ \Box(r \to \Diamond t)\end{array}}{\Box(p \to \Diamond q)} \qquad\qquad \frac{\begin{array}{l}\Box(p \to \Diamond r) \\ \Box(r \to \Diamond q)\end{array}}{\Box(p \to \Diamond q)}$$

These rules belong to the part for proving general temporal validity. They are convenient, but not necessary when we have a relatively complete rule that reduce program validity directly to assertional validity.

Rules for Response Properties (cont.)

A *ranking function* maps finite sequences of states into a well-founded set.

Rule W-RESP (with a ranking function δ)

$$\begin{array}{ll} \text{W1.} & \Box(p \rightarrow (q \vee \varphi)) \\ \text{W2.} & \Box([\varphi \wedge (\delta = \alpha)] \rightarrow \Diamond[q \vee (\varphi \wedge \delta \prec \alpha)]) \\ \hline & \Box(p \rightarrow \Diamond q) \end{array}$$

Rules for Response Properties (cont.)

Let $\mathcal{T} = \{\tau_1, \cdots, \tau_n\}$. φ denotes $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$ and δ is a ranking function.

Rule F-RESP

$$\begin{array}{ll}
\text{F1.} & \Box(p \rightarrow (q \vee \varphi)) \\
\multicolumn{2}{l}{\text{for } i = 1, \cdots, m} \\
\text{F2.} & \{\varphi_i \wedge (\delta = \alpha)\}\ \mathcal{T}\ \{q \vee (\varphi \wedge (\delta \prec \alpha)) \vee (\varphi_i \wedge (\delta \preceq \alpha))\} \\
\text{F3.} & \{\varphi_i \wedge (\delta = \alpha)\}\ \tau_i\ \{q \vee (\varphi \wedge (\delta \prec \alpha))\} \\
\text{J4.} & \Box(\varphi_i \rightarrow (q \vee En(\tau_i))), \text{if } \tau_i \in \mathcal{J} \\
\text{C4.} & \mathcal{T} - \{\tau_i\} \vdash \Box(\varphi_i \rightarrow \Diamond(q \vee En(\tau_i))), \text{if } \tau_i \in \mathcal{C} \\
\hline
 & \Box(p \rightarrow \Diamond q)
\end{array}$$

Rule F-RESP is (relatively) complete for proving the $\mathcal{P}$-validity of any response formula of the form $\Box(p \rightarrow \Diamond q)$.

Rules for Reactivity Properties

Rule B-REAC

$$\begin{array}{ll}
\text{B1.} & \Box(p \rightarrow (q \vee \varphi)) \\
\text{B2.} & \{\varphi \wedge (\delta = \alpha)\}\ \mathcal{T}\ \{q \vee (\varphi \wedge (\delta \preceq \alpha))\} \\
\text{B3.} & \Box([\varphi \wedge (\delta = \alpha) \wedge r] \rightarrow \Diamond[q \vee (\delta \prec \alpha)]) \\
\hline
 & \Box((p \wedge \Box\Diamond r) \rightarrow \Diamond q)
\end{array}$$

For programs without compassionate transitions, Rule B-REAC is (relatively) complete for proving the $\mathcal{P}$-validity of any (simple, extended) reactivity formula of the form $\Box((p \wedge \Box\Diamond r) \rightarrow \Diamond q)$.

Fair Discrete Systems

- An FDS $\mathcal{D}$ is a tuple $\langle V, \Theta, \rho, \mathcal{J}, \mathcal{C} \rangle$:
  - $V \subseteq \mathcal{V}$: A finite set of typed state variables, containing *data* and *control* variables.
  - Θ : The initial condition, an assertion characterizing the initial states.
  - ρ : The transition relation, an assertion relating the values of the state variables in a state to the values in the next state.
  - $\mathcal{J} = \{J_1, \cdots, J_k\}$: A set of justice requirements (weak fairness).
  - $\mathcal{C} = \{\langle p_1, q_1 \rangle, \cdots, \langle p_n, q_n \rangle\}$: A set of compassion requirements (strong fairness).

Fair Discrete Systems (cont.)

- So, FDS is a slight variation of the model of fair transition system.
- The main difference between the FDS and FTS models is in the representation of fairness constraints.
- FDS enables a unified representation of fairness constraints arising from both the system being verified, and the temporal property.
- A computation of $\mathcal{D}$ is an infinite sequence of states $\sigma = s_0, s_1, s_2, \cdots$ satisfying *Initiation*, *Consecution*, *Justice*, and *Compassion* conditions.

Program Mux-Sem as an FDS

- Program MUX-SEM: mutual exclusion by a semaphore.

$$s : \text{natural } \textbf{initially } s = 1$$

$$\left[\begin{array}{l} l_0 : \textbf{loop forever do} \\ \left[\begin{array}{ll} l_1 : & \text{remainder;} \\ l_2 : & \text{request}(s); \\ l_3 : & \text{critical;} \\ l_4 : & \text{release}(s); \end{array}\right] \end{array}\right] \quad \| \quad \left[\begin{array}{l} m_0 : \textbf{loop forever do} \\ \left[\begin{array}{ll} m_1 : & \text{remainder;} \\ m_2 : & \text{request}(s); \\ m_3 : & \text{critical;} \\ m_4 : & \text{release}(s); \end{array}\right] \end{array}\right]$$

  - $\text{request}(s) \triangleq \langle \textbf{await } s > 0 : s := s - 1 \rangle$
  - $\text{release}(s) \triangleq s := s + 1$
- $\mathcal{C}$: $\{(at\_l_2 \wedge s > 0, at\_l_3), (at\_m_2 \wedge s > 0, at\_m_3)\}$