Software Specification and Verification

Course Introduction: Reasoning about Programs

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The Coffee Can Problem

- The Setting:
  - **Initially**: a coffee can contains some black beans and some white beans.
  - **Action**: the following steps are repeated as many times as possible.
    1. Pick any two beans from the can.
    2. If they have the same color, put another black bean in and throw anything else away. (Assume there is a sufficient supply of additional black beans.)
    3. Otherwise, put the white bean back in and throw the black one away.
  - **Finally**: only one bean remains in the can.

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    3. Otherwise, put the white bean back in and throw the black one away.
  - **Finally**: only one bean remains in the can.
- Question: what can be said about the color of the last remaining bean?

The Coffee Can Problem as a Program

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$B, W := m, n;$ // $m > 0 \wedge n > 0$
do $B \geq 0 \wedge W \geq 2 \rightarrow B, W := B + 1, W - 2$ // both white
$[\!]$ $B \geq 2 \wedge W \geq 0 \rightarrow B, W := B - 1, W$ // both black
$[\!]$ $B \geq 1 \wedge W \geq 1 \rightarrow B, W := B - 1, W$ // different colors
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(Note: one of the three alternatives in the **do** loop is arbitrarily chosen and executed until none is "enabled", at which time the loop terminates.)

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- What are the values of B and W, when the program terminates?
- Will the program actually terminate?

Invariants and Rank Functions

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- For the Coffee Can problem,
  - (Loop) Invariant: the parity of the number of white beans never changes, i.e., $W \equiv n \pmod 2$. (in addition, $B + W \geq 1$)
  - Rank Function: the total number of beans, i.e., $B + W$.
  - The do loop decrements the rank function by one in each iteration and eventually terminates when $B + W = 1$ (i.e., $B = 0 \wedge W = 1$ or $B = 1 \wedge W = 0$).

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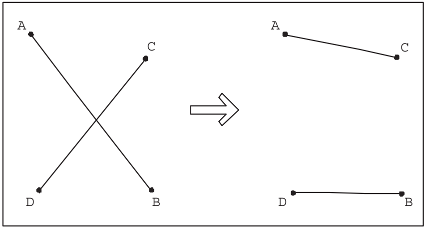
  - The do loop decrements the rank function by one in each iteration and eventually terminates when $B + W = 1$ (i.e., $B = 0 \wedge W = 1$ or $B = 1 \wedge W = 0$).
  - So, what is the color of the remaining bean?

Another Example: Untangling Line Segments

- The Setting:
  - **Initially**: there are $2n$ points on the Euclidean plane. The points are grouped in pairs with a line segment connecting each pair.
  - **Action**: the following untangling operation is repeatedly applied to the points.



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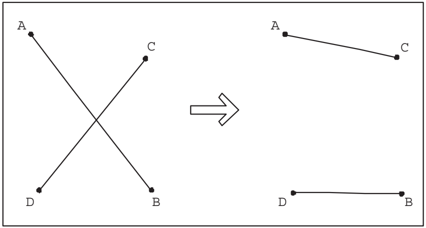

Note that new pairs of crossed line segments may result from this operation.

- Question: will this process terminate?

Untangling Line Segments (cont.)

- **Rank Function**: the total length of all line segments. (Note: this needs to be refined.)
- Each application of the untangling operation reduces the total length (thanks to the triangular inequality).
- The above reduction in length must be greater than some positive constant which is determined in the initial state (by considering all possible groupings of four points).
- The total length is finite and an infinite number of reductions by a positive constant is not possible.
- Therefore, the untangling process will terminate.

Proving Termination Can Be Very Hard

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function collatz(n): integer;
begin
    while $n > 1$ **do**
        if n is even **then** $n := n/2$
        else $n := 3n + 1$
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    return n
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- What would be a suitable rank function for the while loop?

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Note: if the Collatz conjecture is correct, the program will terminate.

References

- I learned the Coffee Can and the Untangling Line Segments problems from Dijkstra's lectures.
- The Coffee Can Problem also appeared in:
  D. Gries. *The Science of Programming*, Springer-Verlag, 1981.
- The Collatz Conjecture:
  https://en.wikipedia.org/wiki/Collatz_conjecture