Final

Note

This is a take-home exam. You may consult any books, papers, or notes, but discussion with other students or seeking outside help is strictly forbidden. Please use A4 paper for writing up your solutions. Drop your solutions in Yih-Kuen Tsay's mail box or email him an electronic version by the deadline.

Problems

1. A majority of an array of n elements is an element that has more than $\frac{n}{2}$ occurrences in the array. Below is a program that finds the majority of an array X of n elements or determines its non-existence. (Hint: if $A[i] \neq A[j]$, then the majority of A remains a majority in a new array B obtained from A by removing $A[i]$ and $A[j]$. Check out Udi Manber's algorithms book if you cannot understand the program.)

```
C,M := X[1],1;
i := 2;
while i<=n do
   if M=0 then C,M := X[i],1
          else if C=X[i] then M := M+1
                         else M := M-1
               fi
   fi;
   i := i+1
od;
if M=0 then Majority := -1
       else Count := 0;
            i := 1;
            while i<=n do
               if X[i]=C then Count := Count+1 fi;
               i := i+1
            od;
            if Count>n/2 then Majority := C
                         else Majority := -1
            fi
fi
```

(a) (20 %) Annotate the program into a proof outline, which shows clearly the correctness of the program.

(b) (10 %) Prove the validity of the annotation. It suffices to elaborate only on the more interesting steps (up to your judgment).

2. The following fundamental properties are usually taken as axioms for the predicate transformer *wp* (weakest precondition):

- **Law of the Excluded Miracle**: $wp(S, \mathit{false}) \equiv \mathit{false}$.
- **Distributivity of Conjunction**: $wp(S, Q_1) \wedge wp(S, Q_2) \equiv wp(S, Q_1 \wedge Q_2)$.
- **Distributivity of Disjunction** for deterministic S: $wp(S, Q_1) \vee wp(S, Q_2) \equiv wp(S, Q_1 \vee Q_2)$.

From the axioms (plus the usual logical and algebraic laws), derive the following properties of *wp* (Hint: not every axiom is useful):

- (10 %) **Law of Monotonicity**: if $Q_1 \Rightarrow Q_2$, then $wp(S, Q_1) \Rightarrow wp(S, Q_2)$.
- (10 %) **Distributivity of Disjunction** (for any command): $wp(S, Q_1) \vee wp(S, Q_2) \Rightarrow wp(S, Q_1 \vee Q_2)$.

3. Solve the following problems for fair transition systems, which we have studied as a model for concurrent reactive systems. You may consider only justice, and ignore compassion, constraints.

(a) (20 %) Give a suitable formal definition for *open* fair transition systems, or fair transition modules, where the set of variables is partitioned into *in* and *out* variables. A system reads from, but does not write on, its *in* variables. The environment of an open system reads from, but does not write on, the *out* variables of the system. The computation of an open system should take into account the interference from its environment.

(b) (10 %) Define a parallel composition operation "$\|$" on two open fair transition systems that follows the interleaving model of concurrency. The parallel composition of two open systems is another open system. Be careful about the condition under which two systems may be composed.

(c) (20 %) For two systems S_1 and S_2 that are composable, prove that the set of computations of $S_1 \| S_2$, namely $\mathit{Comp}(S_1 \| S_2)$, is the intersection of $\mathit{Comp}(S_1)$ and $\mathit{Comp}(S_2)$. (Note: adjust your definitions in the preceding sub-problems so that this compositional property holds.)