

# Theory of Computing

## Introduction and Preliminaries

(Based on [Sipser 2006, 2013])

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- 🌐 The central question:

*What are the fundamental capabilities and limitations of computers?*

- 🌐 Three main areas:





- ☀ *Automata Theory*
- ☀ *Computability Theory*
- ☀ *Complexity Theory*

# Complexity Theory

- Some problems are easy and some hard.  
For example, sorting is easy and scheduling is hard.
- The central question of complexity theory:  
*What makes some problems computationally hard and others easy?*
- We don't have the answer to it.
- However, researchers have found a scheme for **classifying** problems according to their computational difficulty.
- One practical application: cryptography/security.

# Dealing with Hard Problems

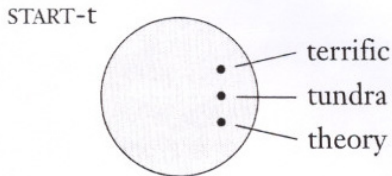
Options for dealing with a hard problem:

-  Try to simplify it (the hard part of the problem might be unnecessary).
-  Settle for an approximate solution.
-  Find a solution that usually runs fast.
-  Consider alternative types of computation.

- 🌐 **Alan Turing**, among other mathematicians, discovered in the 1930s that certain basic problems cannot be solved by computers.
- 🌐 One example is the problem of determining whether a mathematical statement is true or false.
- 🌐 Theoretical models of computers developed at that time eventually lead to the construction of actual computers.
- 🌐 The theories of computability and complexity are closely related.
- 🌐 *Complexity theory* seeks to classify problems as easy ones and hard ones, while in *computability theory* the classification is by whether the problem is solvable or not.

- 🌐 The theories of computability and complexity require a **precise, formal definition** of a *computer*.
- 🌐 *Automata theory* deals with the definitions and properties of mathematical models of computation.
- 🌐 Two basic and practically useful models:
  - ☀️ *Finite-state*, or simply *finite*, *automaton*
  - ☀️ *Context-free grammar* (pushdown automaton)

- 🌐 Set, element (member), subset, proper subset
- 🌐 Multiset
- 🌐 Description of a set
- 🌐 The empty set ( $\emptyset$ )
- 🌐 Finite set, infinite set
- 🌐 Union, intersection, complement
- 🌐 Power set
- 🌐 Venn diagram

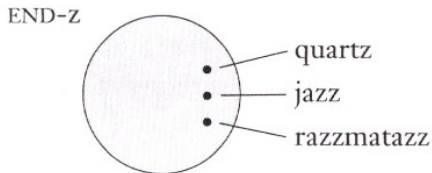


**FIGURE 0.1**

Venn diagram for the set of English words starting with “t”

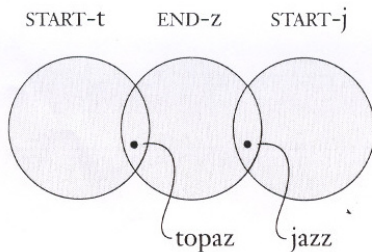
Source: [Sipser 2006]





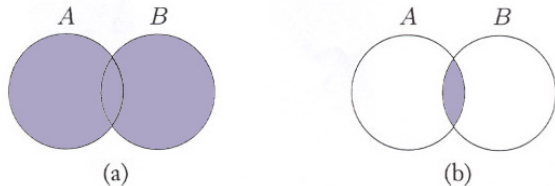
**FIGURE 0.2**  
Venn diagram for the set of English words ending with “z”

Source: [Sipser 2006]



**FIGURE 0.3**  
Overlapping circles indicate common elements

Source: [Sipser 2006]



**FIGURE 0.4**  
Diagrams for (a)  $A \cup B$  and (b)  $A \cap B$

Source: [Sipser 2006]

# Sequences and Tuples

- 🌐 A *sequence* of objects is a list of these objects in some order. Order is essential and repetition is also allowed.
- 🌐 Finite sequences are often called *tuples*. A sequence with  $k$  elements is a  $k$ -tuple; a 2-tuple is also called a *pair*.
- 🌐 The *Cartesian product*, or cross product, of  $A$  and  $B$ , written as  $A \times B$ , is the set of all pairs  $(x, y)$  such that  $x \in A$  and  $y \in B$ .
- 🌐 Cartesian products generalize to  $k$  sets,  $A_1, A_2, \dots, A_k$ , written as  $A_1 \times A_2 \times \dots \times A_k$ .  $A^k$  is a shorthand for  $A \times A \times \dots \times A$  ( $k$  times).

- 🌐 A *function* sets up an *input-output* relationship, where the same input always produces the same output.
- 🌐 If  $f$  is a function whose output is  $b$  when the input is  $a$ , we write  $f(a) = b$ .
- 🌐 A function is also called a *mapping*; if  $f(a) = b$ , we say that  $f$  maps  $a$  to  $b$ .

# Functions (cont.)

- 🌐 The set of possible inputs to a function is called its *domain*; the outputs come from a set called its *range*.
- 🌐 A function is *onto* if it uses all the elements of the range (it is *one-to-one* if ...).
- 🌐 The notation  $f : D \longrightarrow R$  says that  $f$  is a function with domain  $D$  and range  $R$ .
- 🌐 More notions and terms: *k-ary function*, *unary function*, *binary function*, *infix notation*, *prefix notation*

# Relations

- 🌐 A *predicate*, or property, is a function whose range is  $\{\text{TRUE}, \text{FALSE}\}$ .
- 🌐 A predicate whose domain is a set of  $k$ -tuples  $A \times \dots \times A$  is called a ( $k$ -ary) *relation* on  $A$ .
- 🌐 A 2-ary relation is also called a *binary relation*.

# Equivalence Relations

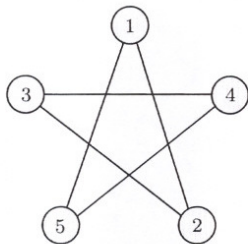
- 🌐 An *equivalence relation* is a special type of binary relation that captures the notion of two objects being *equal* in some sense.
- 🌐 A binary relation  $R$  on  $A$  is an equivalence relation if
  1.  $R$  is *reflexive* (for every  $x$  in  $A$ ,  $xRx$ ),
  2.  $R$  is *symmetric* (for every  $x$  and  $y$  in  $A$ ,  $xRy$  if and only if  $yRx$ ), and
  3.  $R$  is *transitive* (for every  $x$ ,  $y$ , and  $z$  in  $A$ ,  $xRy$  and  $yRz$  implies  $xRz$ ).



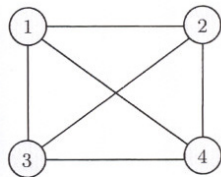
# Graphs

- 🌐 Undirected graph, node (vertex), edge (link), degree
- 🌐 Description of a graph:  $G = (V, E)$
- 🌐 Labeled graph
- 🌐 Subgraph, induced subgraph
- 🌐 Path, simple path, cycle, simple cycle
- 🌐 Connected graph
- 🌐 Tree, root, leaf
- 🌐 Directed graph, outdegree, indegree
- 🌐 Strongly connected graph

# Graphs (cont.)



(a)

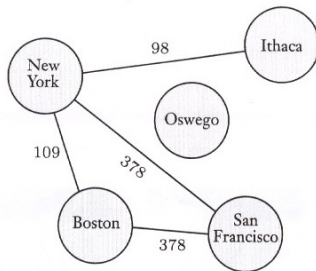


(b)

**FIGURE 0.12**  
Examples of graphs

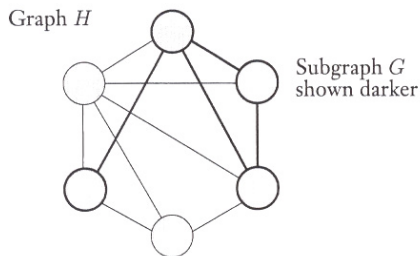
Source: [Sipser 2006]

# Graphs (cont.)



**FIGURE 0.13**  
Cheapest nonstop air fares between various cities

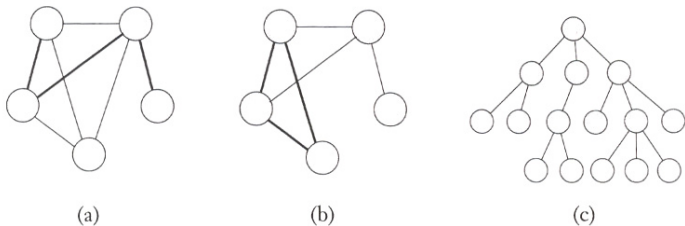
Source: [Sipser 2006]



**FIGURE 0.14**  
Graph  $G$  (shown darker) is a subgraph of  $H$

Source: [Sipser 2006]

# Graphs (cont.)

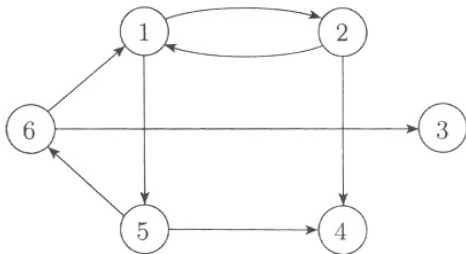


**FIGURE 0.15**

(a) A path in a graph, (b) a cycle in a graph, and (c) a tree

Source: [Sipser 2006]

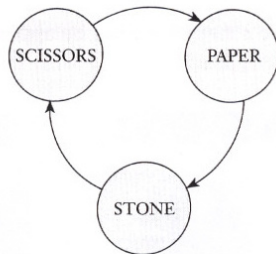
# Graphs (cont.)



**FIGURE 0.16**  
A directed graph

Source: [Sipser 2006]

# Graphs (cont.)



**FIGURE 0.18**  
The graph of the relation *beats*



Source: [Sipser 2006]

# Strings and Languages

- 🌐 An *alphabet* is any finite set of *symbols*.
- 🌐 A *string* over an alphabet is a finite sequence of symbols from that alphabet.
- 🌐 The *length* of a string  $w$ , written as  $|w|$ , is the number of symbols that  $w$  contains.
- 🌐 The string of length 0 is called the *empty string*, written as  $\varepsilon$ .
- 🌐 The *concatenation* of  $x$  and  $y$ , written as  $xy$ , is the string obtained from appending  $y$  to the end of  $x$ .
- 🌐 A *language* is a set of strings.
- 🌐 More notions and terms: *reverse*, *substring*, *lexicographic ordering*.



# Boolean Logic

-  *Boolean logic* is a mathematical system built around the two *Boolean values* TRUE (1) and FALSE (0).
-  Boolean values can be manipulated with *Boolean operations*: *negation* or NOT ( $\neg$ ), *conjunction* or AND ( $\wedge$ ), *disjunction* or OR ( $\vee$ ).

$$0 \wedge 0 \triangleq 0$$

$$0 \vee 0 \triangleq 0$$

$$\neg 0 \triangleq 1$$

$$0 \wedge 1 \triangleq 0$$

$$0 \vee 1 \triangleq 1$$


$$\neg 1 \triangleq 0$$

$$1 \wedge 0 \triangleq 0$$

$$1 \vee 0 \triangleq 1$$

$$1 \wedge 1 \triangleq 1$$

$$1 \vee 1 \triangleq 1$$

-  Unknown Boolean values are represented symbolically by *Boolean variables or propositions*, e.g.,  $P$ ,  $Q$ , etc.

# Boolean Logic (cont.)

- 🌐 Additional Boolean operations: *exclusive or* or XOR ( $\oplus$ ), *equality/equivalence* ( $\leftrightarrow$  or  $\equiv$ ), *implication* ( $\rightarrow$ ).

$$0 \oplus 0 \triangleq 0$$

$$0 \leftrightarrow 0 \triangleq 1$$

$$0 \rightarrow 0 \triangleq 1$$

$$0 \oplus 1 \triangleq 1$$

$$0 \leftrightarrow 1 \triangleq 0$$

$$0 \rightarrow 1 \triangleq 1$$

$$1 \oplus 0 \triangleq 1$$

$$1 \leftrightarrow 0 \triangleq 0$$

$$1 \rightarrow 0 \triangleq 0$$

$$1 \oplus 1 \triangleq 0$$

$$1 \leftrightarrow 1 \triangleq 1$$

$$1 \rightarrow 1 \triangleq 1$$

- 🌐 All in terms of conjunction and negation:

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

$$P \rightarrow Q \equiv \neg P \vee Q$$

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$

$$P \oplus Q \equiv \neg(P \leftrightarrow Q)$$

# Logical Equivalences and Laws

- 🌐 Two logical expressions/formulae are *equivalent* if each of them implies the other, i.e., they have the same truth value.
- 🌐 Equivalence plays a role analogous to equality in algebra.
- 🌐 Some laws of Boolean logic:
  - ☀ (Distributive)  $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
  - ☀ (Distributive)  $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  - ☀ (De Morgan's)  $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
  - ☀ (De Morgan's)  $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

# Definitions, Theorems, and Proofs

- 🌐 *Definitions* describe the objects and notions that we use. Precision is essential to any definition.
- 🌐 After we have defined various objects and notions, we usually make *mathematical statements* about them. Again, the statements must be precise.
- 🌐 A *proof* is a convincing logical argument that a statement is true. The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof.
- 🌐 A *theorem* is a mathematical statement proven true. *Lemmas* are proven statements for assisting the proof of another more significant statement.
- 🌐 *Corollaries* are statements seen to follow easily from other proven ones.

# Finding Proofs

🌐 Find proofs isn't always easy; no one has a recipe for it.

🌐 Below are some helpful general strategies:

1. Carefully read the statement you want to prove.
2. Rewrite the statement in your own words.
3. Break it down and consider each part separately.  
For example,  $P \iff Q$  consists of two parts:  $P \rightarrow Q$  (the forward direction) and  $Q \rightarrow P$  (the reverse direction).
4. Try to get an intuitive feeling of why it should be true.

# Tips for Producing a Proof

- 🌐 A well-written proof is a sequence of statements, wherein each one follows by simple reasoning from previous statements in the sequence.
- 🌐 Tips for producing a proof:
  - ☀️ *Be patient.* Finding proofs takes time.
  - ☀️ *Come back to it.* Look over the statement, think about it, leave it, and then return some time later.
  - ☀️ *Be neat.* Use simple, clear text and/or pictures; make it easy for others to understand.
  - ☀️ *Be concise.* Emphasize high-level ideas, but be sure to include enough details of reasoning.

# An Example Proof

## Theorem

For any two sets  $A$  and  $B$ ,  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ .

Proof. We show that every element of  $\overline{A \cup B}$  is also an element of  $\overline{A} \cap \overline{B}$  and vice versa.

Forward ( $x \in \overline{A \cup B} \rightarrow x \in \overline{A} \cap \overline{B}$ ):

- $x \in \overline{A \cup B}$
- $\rightarrow x \notin A \cup B$  , def. of complement
- $\rightarrow x \notin A$  and  $x \notin B$  , def. of union
- $\rightarrow x \in \overline{A}$  and  $x \in \overline{B}$  , def. of complement
- $\rightarrow x \in \overline{A} \cap \overline{B}$  , def. of intersection

Reverse ( $x \in \overline{A} \cap \overline{B} \rightarrow x \in \overline{A \cup B}$ ): ...

# Another Example Proof

## Theorem

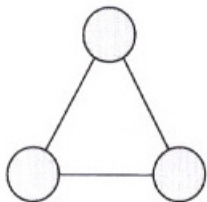
*In any graph  $G$ , the sum of the degrees of the nodes of  $G$  is an even number.*

Proof.

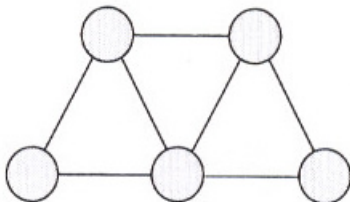
- 🌐 Every edge in  $G$  connects two nodes, contributing 1 to the degree of each.
- 🌐 Therefore, each edge contributes 2 to the sum of the degrees of all the nodes.
- 🌐 If  $G$  has  $e$  edges, then the sum of the degrees of the nodes of  $G$  is  $2e$ , which is even.



# Another Example Proof (cont.)



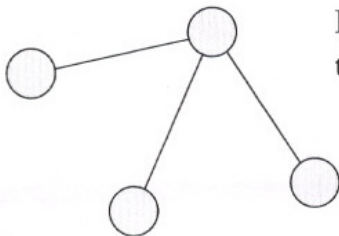
$$\begin{aligned}\text{sum} &= 2+2+2 \\ &= 6\end{aligned}$$



$$\begin{aligned}\text{sum} &= 2+3+4+3+2 \\ &= 14\end{aligned}$$

Source: [Sipser 2006]




## Another Example Proof (cont.)



Every time an edge is added,  
the sum increases by 2.

Source: [Sipser 2006]

# Types of Proof

-  *Proof by construction:*  
prove that a particular type of object exists, by showing how to construct the object.
-  *Proof by contradiction:*  
prove a statement by first assuming that the statement is false and then showing that the assumption leads to an obviously false consequence, called a contradiction.
-  *Proof by induction:*  
prove that all elements of an infinite set have a specified property, by exploiting the inductive structure of the set.

# Proof by Construction

## Theorem

*For each even number  $n$  greater than 2, there exists a 3-regular graph with  $n$  nodes.*

Proof. Construct a graph  $G = (V, E)$  with  $n$  ( $= 2k \geq 2$ ) nodes as follows.

Let  $V$  be  $\{0, 1, \dots, n - 1\}$  and  $E$  be defined as

$$E = \{ \{i, i + 1\} \mid \text{for } 0 \leq i \leq n - 2 \} \cup \\ \{ \{n - 1, 0\} \} \cup \\ \{ \{i, i + n/2\} \mid \text{for } 0 \leq i \leq n/2 - 1 \}.$$

# Proof by Contradiction

## Theorem

$\sqrt{2}$  is irrational.

Proof. Assume toward a contradiction that  $\sqrt{2}$  is rational, i.e.,  $\sqrt{2} = \frac{m}{n}$  for some integers  $m$  and  $n$ , which *cannot both be even*.

$\sqrt{2} = \frac{m}{n}$	, from the assumption
$n\sqrt{2} = m$	, multipl. both sides by $n$
$2n^2 = m^2$	, square both sides
$m$ is even	, $m^2$ is even
$2n^2 = (2k)^2 = 4k^2$	, from the above two
$n^2 = 2k^2$	, divide both sides by 2
$n$ is even	, $n^2$ is even

Now both  $m$  and  $n$  are even, a contradiction.

## Example: Home Mortgages

$P$ : the *principle* (amount of the original loan).

$I$ : the yearly *interest rate*.

$Y$ : the monthly payment.

$M$ : the *monthly multiplier*  $= 1 + I/12$ .

$P_t$ : the amount of loan outstanding after the  $t$ -th month;  $P_0 = P$   
and  $P_{k+1} = P_k M - Y$ .

### Theorem

For each  $t \geq 0$ ,

$$P_t = PM^t - Y \left( \frac{M^t - 1}{M - 1} \right).$$

# Proof by Induction

## Theorem

For each  $t \geq 0$ ,

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right).$$

Proof. The proof is by induction on  $t$ .

 *Basis:* When  $t = 0$ ,  $PM^0 - Y\left(\frac{M^0 - 1}{M - 1}\right) = P = P_0$ .

# Proof by Induction (cont.)

 *Induction step:* When  $t = k + 1$  ( $k \geq 0$ ),

$$\begin{aligned} & P_{k+1} \\ = & \quad \{\text{definition of } P_t\} \\ & P_k M - Y \\ = & \quad \{\text{the induction hypothesis}\} \\ & (PM^k - Y(\frac{M^k - 1}{M - 1}))M - Y \\ = & \quad \{\text{distribute } M \text{ and rewrite } Y\} \\ & PM^{k+1} - Y(\frac{M^{k+1} - M}{M - 1}) - Y(\frac{M - 1}{M - 1}) \\ = & \quad \{\text{combine the last two terms}\} \\ & PM^{k+1} - Y(\frac{M^{k+1} - 1}{M - 1}) \end{aligned}$$