

# Theory of Computing 2021: Decidability

(Based on [Sipser 2006, 2013])

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## 1 Introduction

### Decidability/Solvability

- We shall demonstrate certain problems that can be solved algorithmically and others that cannot.
- Our objective is to explore the limits of algorithmic solvability.
- Why should you study unsolvability?
  - Knowing when a problem is algorithmically unsolvable is useful because then you realize that the problem must be simplified or altered before you can find an algorithmic solution.
  - A glimpse of the unsolvable can stimulate your imagination and help you gain an important perspective on computation.

## 2 Decidable Languages

### Decidable Languages/Problems

- $A_{\text{DFA}} = \{\langle B, w \rangle \mid B \text{ is a DFA that accepts } w\}$ .
- This is the *acceptance problem* (membership problem) for DFAs formulated as a language.

**Theorem 1** (4.1).  $A_{\text{DFA}}$  is a decidable language.

- $M =$  “On input  $\langle B, w \rangle$ , where  $B$  is a DFA and  $w$  is a string:
  1. Simulate  $B$  on input  $w$ .
  2. If the simulation ends in an accept state, *accept*; otherwise, reject.”

### Decidable Languages/Problems (cont.)

- $A_{\text{NFA}} = \{\langle B, w \rangle \mid B \text{ is an NFA that accepts } w\}$ .

**Theorem 2** (4.2).  $A_{\text{NFA}}$  is a decidable language.

- $N =$  “On input  $\langle B, w \rangle$ , where  $B$  is an NFA and  $w$  is a string:
  1. Convert NFA  $B$  to an equivalent DFA  $C$ .
  2. Run TM  $M$  for deciding  $A_{\text{DFA}}$  (as a “procedure”) on input  $\langle C, w \rangle$ .
  3. If  $M$  accepts, *accept*; otherwise, reject.”

### Decidable Languages/Problems (cont.)

- $A_{\text{REX}} = \{\langle R, w \rangle \mid R \text{ is a regular expression that generates } w\}$ .

**Theorem 3** (4.3).  $A_{\text{REX}}$  is a decidable language.

- $P =$  “On input  $\langle R, w \rangle$ , where  $R$  is a regular expression and  $w$  is a string:
  1. Convert regular expression  $R$  to an equivalent DFA  $A$ .
  2. Run TM  $M$  for deciding  $A_{\text{DFA}}$  on input  $\langle A, w \rangle$ .
  3. If  $M$  accepts, *accept*; otherwise, reject.”

### Decidable Languages/Problems (cont.)

- $E_{\text{DFA}} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$ .

**Theorem 4** (4.4).  $E_{\text{DFA}}$  is a decidable language.

- $T =$  “On input  $\langle A \rangle$ , where  $A$  is a DFA:
  1. Mark the start state of  $A$ .
  2. Repeat Step 3 until no new states get marked.
  3. Mark any state that has a transition coming into it from any state that is already marked.
  4. If no accept state is marked, *accept*; otherwise, reject.”

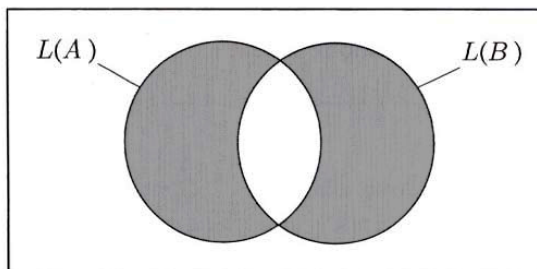
### Decidable Languages/Problems (cont.)

- $EQ_{\text{DFA}} = \{\langle A, B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$ .

**Theorem 5** (4.5).  $EQ_{\text{DFA}}$  is a decidable language.

- $F =$  “On input  $\langle A, B \rangle$ , where  $A$  and  $B$  are DFAs:
  1. Construct DFA  $C = (A \cap \bar{B}) \cup (\bar{A} \cap B)$ .
  2. Run TM  $T$  for deciding  $E_{\text{DFA}}$  on input  $\langle C \rangle$ .
  3. If  $T$  accepts, *accept*; otherwise, reject.”

### Decidable Languages/Problems (cont.)



**FIGURE 4.6**  
The symmetric difference of  $L(A)$  and  $L(B)$

Source: [Sipser 2006]

## Decidable CFL Properties

- $A_{CFG} = \{\langle G, w \rangle \mid G \text{ is a CFG that generates } w\}$ .

**Theorem 6 (4.7).**  $A_{CFG}$  is a decidable language.

- $S =$  “On input  $\langle G, w \rangle$ , where  $G$  is a CFG and  $w$  is a string:
  1. Convert  $G$  to an equivalent grammar in Chomsky normal form.
  2. List all derivations with  $2|w| - 1$  steps.
  3. If any of these derivations generate  $w$ , *accept*; otherwise, reject.”

## Decidable CFL Properties (cont.)

- $E_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ .

**Theorem 7 (4.8).**  $E_{CFG}$  is a decidable language.

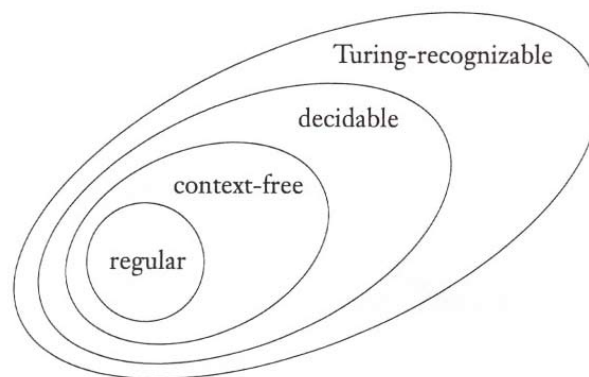
- $R =$  “On input  $\langle G \rangle$ , where  $G$  is a CFG:
  1. Mark all terminals in  $G$ .
  2. Repeat Step 3 until no new variables get marked.
  3. Mark any variable  $A$  where  $A \rightarrow U_1 U_2 \cdots U_k$  is a rule in  $G$  and each symbol  $U_1, U_2, \dots, U_k$  has already been marked.
  4. If the start symbol is not marked, *accept*; otherwise, reject.”

## Decidability of CFLs

**Theorem 8 (4.9).** Every context-free language is decidable.

- Let  $G$  be a CFG for the given language  $A$  and design a TM  $M_G$  that decides  $A$ .
- $M_G =$  “On input  $w$ :
  1. Run TM  $S$  for deciding  $A_{CFG}$  on input  $\langle G, w \rangle$ .
  2. If  $S$  accepts, *accept*; otherwise, reject.”

## Classes of Languages



**FIGURE 4.10**  
The relationship among classes of languages

Source: [Sipser 2006]

## Classes of Languages (cont.)

Chomsky Hierarchy	Grammar	Language	Computation Model
Type-0	Unrestricted	R.E.	Turing Machine
N/A	(no common name)	Recursive	Decider
Type-1	Context-Sensitive	Context-Sensitive	Linear Bounded
Type-2	Context-Free	Context-Free	Pushdown
Type-3	Regular	Regular	Finite

- Recall that Recursively Enumerable (R.E.)  $\equiv$  Turing-recognizable and Recursive  $\equiv$  Decidable (Turing-decidable).
- Linear Bounded Automata will be introduced later.

## 3 The Halting Problem

### Undecidability

- We shall prove that *there is a specific problem that is algorithmically unsolvable.*
- This result demonstrates that computers are limited in a very fundamental way.
- Unsolvable problems are not necessarily esoteric. Some ordinary problems that people want to solve may turn out to be unsolvable.
- For example, the general problem of software verification is not solvable by computer.
- The specific problem that we will prove algorithmically unsolvable is the one of *testing whether a Turing machine accepts a given input string.*

### The Acceptance Problem

- $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ .

**Theorem 9** (4.11).  $A_{TM}$  is undecidable.

- We will prove this fundamental result later.
- On the other hand,  $A_{TM}$  is Turing-recognizable.

### The Acceptance Problem (cont.)

- $U =$  “On input  $\langle M, w \rangle$ , where  $M$  is a TM and  $w$  is a string:
  1. Simulate  $M$  on input  $w$ .
  2. If  $M$  ever enters its accept state, *accept*; if  $M$  ever enters its reject state, reject.”
- If we had (actually not) some way to determine that  $M$  was not *halting* on  $w$ , then we could turn the recognizer  $U$  into a decider.

Note: The Turing machine  $U$  is an example of the *universal Turing machine*, as it is capable of simulating any other Turing machine from the description of that machine. The universal Turing machine inspired “stored-program” computers.

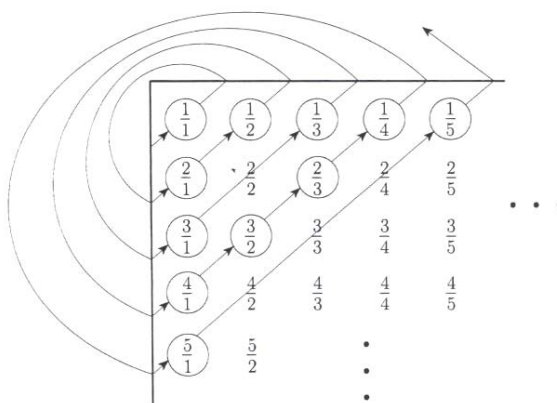
## Countable vs. Uncountable Sets

**Definition 10** (4.12). Let  $f$  be a function from  $A$  to  $B$ .

- We say that  $f$  is *one-to-one* if  $f(a) \neq f(b)$  whenever  $a \neq b$ .
- Say that  $f$  is *onto* if, for every  $b \in B$ , there is an  $a \in A$  such that  $f(a) = b$ .
- A function that is both one-to-one and onto is called a *correspondence*.
- Two sets are considered to have the same size if there is a correspondence between them.

**Definition 11** (4.14). A set  $A$  is **countable** if either it is finite or it has the same size as  $\mathcal{N} = \{1, 2, 3, \dots\}$ ; it is **uncountable**, otherwise.

## Countable vs. Uncountable Sets (cont.)



**FIGURE 4.16**  
A correspondence of  $\mathcal{N}$  and  $\mathcal{Q}$

Source: [Sipser 2006]

## Uncountable Sets

- A real number is one that has a (possibly infinite) decimal representation.
- Let  $\mathcal{R}$  be the set of real numbers.

**Theorem 12** (4.17).  $\mathcal{R}$  is uncountable.

## Uncountable Sets (cont.)

- Assume that a correspondence  $f$  existed between  $\mathcal{N}$  and  $\mathcal{R}$ .

$n$	$f(n)$
1	3. <u>1</u> 4159...
2	55.5 <u>5</u> 555...
3	0.12 <u>3</u> 45...
4	0.500 <u>0</u> 0...
⋮	⋮

- We can find an  $x$ ,  $0 < x < 1$ , so that the  $i$ -th digit following the decimal point of  $x$  is different from that of  $f(i)$ ; for example,  $x = 0.4641\dots$  is a possible choice.
- This proof technique is called *diagonalization*, discovered by Georg Cantor in 1873.

### Unrecognizability

**Corollary 13** (4.18). *Some languages are not Turing-recognizable.*

- The set of all Turing machines is countable because each Turing machine  $M$  has an encoding into a string  $\langle M \rangle$ .
- Let  $\mathcal{L}$  be the set of all languages over alphabet  $\Sigma$ .
- We can show that there is a correspondence between  $\mathcal{L}$  and the uncountable set  $\mathcal{B}$  of all infinite binary sequences.
  - Let  $\Sigma^* = \{s_1, s_2, s_3, \dots\}$ .
  - Each language  $A \in \mathcal{L}$  has a unique sequence in  $\mathcal{B}$ , where the  $i$ -th bit is a 1 if and only if  $s_i \in A$ .

### Undecidability of the Acceptance Problem

- Suppose  $H$  is a decider for  $A_{\text{TM}}$ :

$$H(\langle M, w \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

- Let  $D =$  “On input  $\langle M \rangle$ , where  $M$  is a TM:
  1. Run  $H$  on input  $\langle M, \langle M \rangle \rangle$ .
  2. If  $H$  accepts, reject and if  $H$  rejects, *accept*.”
- When  $D$  takes itself, namely  $\langle D \rangle$ , as input:

$$D(\langle D \rangle) = \begin{cases} \text{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \text{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

### Undecidability of the Acceptance Problem (cont.)

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\dots$
$M_1$	<i>accept</i>	<i>accept</i>	<i>accept</i>	<i>accept</i>	$\dots$
$M_2$	<i>accept</i>	<i>accept</i>	<i>accept</i>	<i>accept</i>	$\dots$
$M_3$	<i>accept</i>	<i>accept</i>	$\dots$	$\dots$	$\dots$
$M_4$	<i>accept</i>	<i>accept</i>	$\dots$	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

**FIGURE 4.19**  
Entry  $i, j$  is *accept* if  $M_i$  accepts  $\langle M_j \rangle$

Source: [Sipser 2006]

## Undecidability of the Acceptance Problem (cont.)

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\dots$
$M_1$	<i>accept</i>	<i>reject</i>	<i>accept</i>	<i>reject</i>	
$M_2$	<i>accept</i>	<i>accept</i>	<i>accept</i>	<i>accept</i>	$\dots$
$M_3$	<i>reject</i>	<i>reject</i>	<i>reject</i>	<i>reject</i>	
$M_4$	<i>accept</i>	<i>accept</i>	<i>reject</i>	<i>reject</i>	
$\vdots$			$\vdots$		

**FIGURE 4.20**  
Entry  $i, j$  is the value of  $H$  on input  $\langle M_i, \langle M_j \rangle \rangle$

Source: [Sipser 2006]

## Undecidability of the Acceptance Problem (cont.)

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\dots$	$\langle D \rangle$	$\dots$
$M_1$	<u><i>accept</i></u>	<i>reject</i>	<i>accept</i>	<i>reject</i>		<i>accept</i>	
$M_2$	<u><i>accept</i></u>	<u><i>accept</i></u>	<i>accept</i>	<i>accept</i>	$\dots$	<i>accept</i>	$\dots$
$M_3$	<i>reject</i>	<i>reject</i>	<u><i>reject</i></u>	<i>reject</i>	$\dots$	<i>reject</i>	
$M_4$	<i>accept</i>	<i>accept</i>	<i>reject</i>	<u><i>reject</i></u>	$\dots$	<i>accept</i>	
$\vdots$			$\vdots$		$\ddots$		
$D$	<i>reject</i>	<i>reject</i>	<i>accept</i>	<i>accept</i>	$\dots$	<u>?</u>	$\dots$
$\vdots$			$\vdots$		$\ddots$		

**FIGURE 4.21**  
If  $D$  is in the figure, a contradiction occurs at “?”

Source: [Sipser 2006]

## A Turing-Unrecognizable Language

- A language is *co-Turing-recognizable* if it is the complement of a Turing-recognizable language.

**Theorem 14 (4.22).** *A language is decidable if and only if it is both Turing-recognizable and co-Turing-recognizable.*

- Let  $M_1$  be a recognizer for  $A$  and  $M_2$  be a recognizer for  $\bar{A}$ .
- $M =$  “On input  $w$ :
  1. Run both  $M_1$  and  $M_2$  on input  $w$  in parallel. ( $M$  takes turns simulating one step of each machine until one of them halts.)
  2. If  $M_1$  accepts, *accept* and if  $M_2$  accepts, *reject*.”

### A Turing-Unrecognizable Language (cont.)

- $\overline{A_{\text{TM}}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ does not accept } w\}$ .

**Corollary 15** (4.23).  $\overline{A_{\text{TM}}}$  is not Turing-recognizable.

- $A_{\text{TM}}$  is Turing-recognizable, but not decidable.
- From Theorem 4.22,  $A_{\text{TM}}$  must not be co-Turing-recognizable.
- Therefore,  $\overline{A_{\text{TM}}}$  is not Turing-recognizable.