More NP-Complete Problems

(Based on [Sipser 2006, 2013])

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The Vertex Cover Problem

- A *vertex cover* of an undirected graph G is a subset of the nodes where every edge of G touches one of those nodes.
- $VERTEX\_COVER = \{\langle G, k\rangle \mid G$ is an undirected graph that has a k-node vertex cover$\}$.

Theorem

VERTEX_COVER is NP-complete.

- We show that $3SAT \leq_{\mathrm{P}} VERTEX\_COVER$.

The Vertex Cover Problem (cont.)

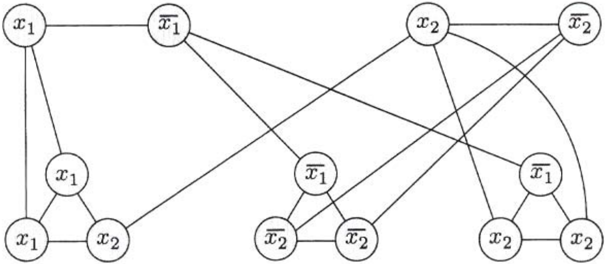

FIGURE 7.45
The graph that the reduction produces from
$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$

Source: [Sipser 2006]

Note: Let k be $m + 2l$, where m is the number of variables and l the number of clauses in ϕ.

The Hamiltonian Path Problem

Theorem

HAMPATH is NP-complete.

We show that $3SAT \leq_{\mathrm{P}} HAMPATH$.

The Hamiltonian Path Problem (cont.)

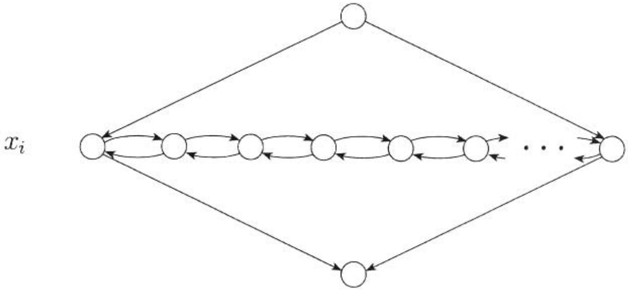

FIGURE 7.47
Representing the variable x_i as a diamond structure

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

FIGURE 7.48
Representing the clause c_j as a node

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

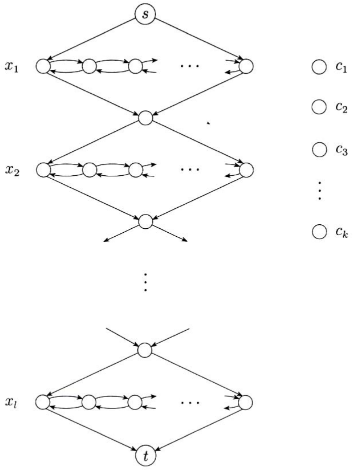

FIGURE 7.49
The high-level structure of G

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

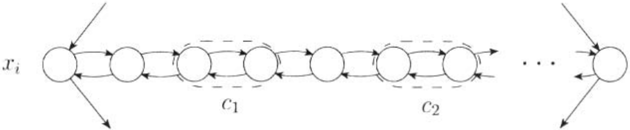

FIGURE 7.50
The horizontal nodes in a diamond structure

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

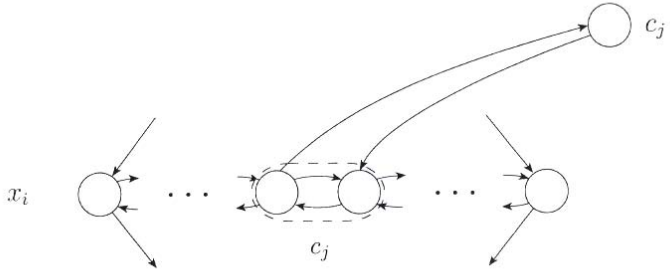

FIGURE 7.51
The additional edges when clause c_j contains x_i

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

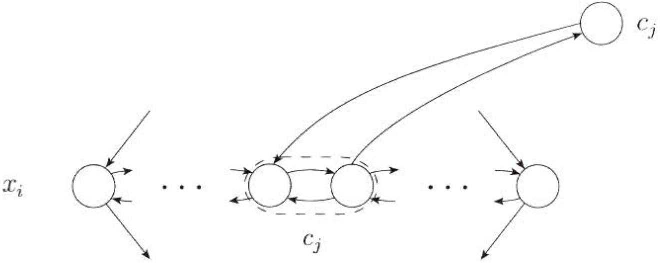

FIGURE 7.52
The additional edges when clause c_j contains $\overline{x_i}$

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

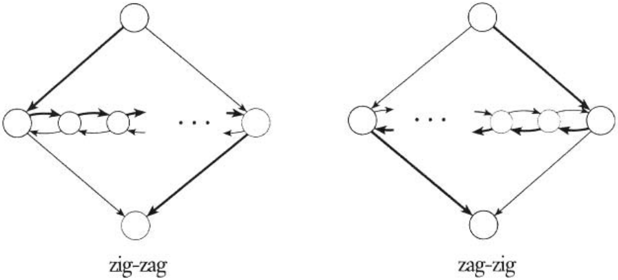

FIGURE 7.53
Zig-zagging and zag-zigging through a diamond, as determined by the satisfying assignment

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

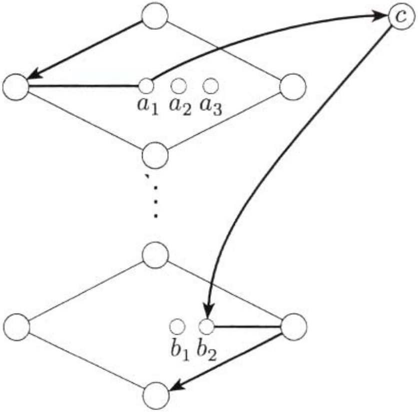

FIGURE **7.54**
This situation cannot occur

Source: [Sipser 2006]

The Hamiltonian Path Problem (cont.)

- Let *UHAMPATH* be the undirected version of the Hamiltonian path problem *HAMPATH*.

Theorem

UHAMPATH is NP-complete.

- An input $\langle G, s, t\rangle$ for *HAMPATH* is mapped to $\langle G', s', t'\rangle$ for *UHAMPATH* as follows.
- Each node u of G, except for s and t, is replaced by a triple of nodes u^{in}, u^{mid}, and u^{out} in G'.
- Nodes s and t are replaced by node $s^{\text{out}} = s'$ and $t^{\text{in}} = t'$.
- Edges connect u^{mid} with u^{in} and u^{out}.
- An edge connects u^{out} and v^{in} if (u, v) is an edge of G.

The Subset Sum Problem

- $SUBSET\_SUM = \{\langle S, t\rangle \mid S = \{x_1, \cdots, x_k\}$ and for some $\{y_1, \cdots, y_l\} \subseteq S$, we have $\sum y_i = t\}$.

Theorem

SUBSET_SUM is NP-complete.

- We show that $3SAT \leq_{\mathrm{P}} SUBSET\_SUM$.

The Subset Sum Problem (cont.)

	1	2	3	4	$\cdots$	l	c_1	c_2	$\cdots$	c_k
y_1	1	0	0	0	$\cdots$	0	1	0	$\cdots$	0
z_1	1	0	0	0	$\cdots$	0	0	0	$\cdots$	0
y_2		1	0	0	$\cdots$	0	0	1	$\cdots$	0
z_2		1	0	0	$\cdots$	0	1	0	$\cdots$	0
y_3			1	0	$\cdots$	0	1	1	$\cdots$	0
z_3			1	0	$\cdots$	0	0	0	$\cdots$	1
$\vdots$					$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_l						1	0	0	$\cdots$	0
z_l						1	0	0	$\cdots$	0
g_1							1	0	$\cdots$	0
h_1							1	0	$\cdots$	0
g_2								1	$\cdots$	0
h_2								1	$\cdots$	0
$\vdots$									$\ddots$	$\vdots$
g_k										1
h_k										1
t	1	1	1	1	$\cdots$	1	3	3	$\cdots$	3

FIGURE 7.57
Reducing *3SAT* to *SUBSET-SUM*

Source: [Sipser 2006]