

# Theory of Computing 2023: Reducibility

(Based on [Sipser 2006, 2013])

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## 1 Introduction

### Introduction

- A *reduction* is a way of converting one problem into another problem in such a way that a solution to the second problem can be used to solve the first problem.
- If a problem  $A$  reduces (is reducible) to another problem  $B$ , we can use a solution to  $B$  to solve  $A$ .
- *Reducibility* says nothing about solving  $A$  or  $B$  alone, but only about the solvability of  $A$  in the presence of a solution to  $B$ .
- Reducibility is the primary method for proving that problems are computationally unsolvable.
- Suppose that  $A$  is reducible to  $B$ . If  $B$  is decidable, then  $A$  is decidable; equivalently, if  $A$  is undecidable, then  $B$  is undecidable.

## 2 Undecidable Problems

### The Halting Problem

- $HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on } w\}$ .

**Theorem 1** (5.1).  $HALT_{TM}$  is undecidable.

- The idea is to reduce the acceptance problem  $A_{TM}$  (shown to be undecidable) to  $HALT_{TM}$ .
- Assume toward a contradiction that a TM  $R$  decides  $HALT_{TM}$ .
- We could then construct a decider  $S$  for  $A_{TM}$  as follows.

### The Halting Problem (cont.)

$S =$  “On input  $\langle M, w \rangle$ , an encoding of a TM  $M$  and a string  $w$ :

1. Run TM  $R$  on input  $\langle M, w \rangle$ .
2. If  $R$  rejects, reject.
3. If  $R$  accepts, simulate  $M$  on  $w$  until it halts.
4. If  $M$  has accepted, *accept*; if  $M$  has rejected, reject.”

### Undecidable Problems

- $E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ .

**Theorem 2 (5.2).**  $E_{\text{TM}}$  is undecidable.

- Assuming that a TM  $R$  decides  $E_{\text{TM}}$ , we construct a decider  $S$  for  $A_{\text{TM}}$  as follows.

### Undecidable Problems (cont.)

$S =$  “On input  $\langle M, w \rangle$ :

1. Construct the following TM  $M_1$ .

$M_1 =$  “On input  $x$ :

(a) If  $x \neq w$ , reject.

(b) If  $x = w$ , run  $M$  on input  $w$  and *accept* if  $M$  accepts  $w$ .”

2. Run  $R$  on input  $\langle M_1 \rangle$ .

3. If  $R$  accepts, reject; if  $R$  rejects, *accept*.”

### Undecidable Problems (cont.)

- $REGULAR_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$ .

**Theorem 3 (5.3).**  $REGULAR_{\text{TM}}$  is undecidable.

- Assuming that a TM  $R$  decides  $REGULAR_{\text{TM}}$ , we construct a decider  $S$  for  $A_{\text{TM}}$  as follows.

### Undecidable Problems (cont.)

$S =$  “On input  $\langle M, w \rangle$ :

1. Construct the following TM  $M_2$ .

$M_2 =$  “On input  $x$ :

(a) If  $x$  has the form  $0^n 1^n$ , *accept*.

(b) If  $x$  does not have this form, run  $M$  on input  $w$  and *accept* if  $M$  accepts  $w$ .”

2. Run  $R$  on input  $\langle M_2 \rangle$ .

3. If  $R$  accepts, *accept*; if  $R$  rejects, reject.”

Note: if  $M$  does not accept  $w$ , then  $L(M_2) = \{0^n 1^n \mid n \geq 0\}$ , which is not regular; if  $M$  accepts  $w$ , then  $L(M_2) = \{0, 1\}^*$ , which is regular.

### Undecidable Problems (cont.)

- $EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ .

**Theorem 4 (5.4).**  $EQ_{\text{TM}}$  is undecidable.

- Assume that a TM  $R$  decides  $EQ_{\text{TM}}$ .

- We construct a decider  $S$  for  $E_{\text{TM}}$  as follows.

- $S =$  “On input  $\langle M \rangle$ :

1. Run  $R$  on input  $\langle M, M_1 \rangle$ , where  $M_1$  is a TM that rejects all inputs.

2. If  $R$  accepts, *accept*; if  $R$  rejects, reject.”

## Rice's Theorem

**Theorem 5.** Any “nontrivial” property about the languages recognized by Turing machines is undecidable.

- Note 1: the theorem considers only properties about languages, i.e., properties that do not distinguish equivalent Turing machine descriptions.
- Note 2: a property is *nontrivial* if it is satisfied by some, but not all, Turing machine descriptions.

## 3 Reduction via Computation Histories

### Computation Histories

**Definition 6** (5.5). An *accepting computation history* for  $M$  on  $w$  is a sequence of configurations  $C_1, C_2, \dots, C_l$ , where

1.  $C_1$  is the start configuration,
2.  $C_l$  is an accepting configuration, and
3.  $C_i$  yields  $C_{i+1}$ ,  $1 \leq i \leq l - 1$ .

A *rejecting computation history* for  $M$  on  $w$  is defined similarly, except that  $C_l$  is a rejecting configuration.

- Computation histories are finite sequences.
- Deterministic machines have at most one computation history on any given input.

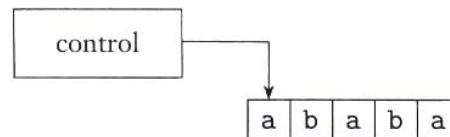
### Linear Bounded Automata

**Definition 7** (5.6). A *linear bounded automaton* (LBA) is a restricted type of Turing machine wherein the tape head is not permitted to move off the portion of the tape containing the input.

- So, an LBA is a TM with a limited amount of memory. It can only solve problems requiring memory that can fit within the tape used for the input.

(Note: using a tape alphabet larger than the input alphabet allows the available memory to be increased up to a constant factor.)

### Linear Bounded Automata (cont.)



**FIGURE 5.7**  
Schematic of a linear bounded automaton

Source: [Sipser 2006]

### Linear Bounded Automata (cont.)

Despite their memory constraint, LBAs are quite powerful.

**Lemma 8** (5.8). *Let  $M$  be an LBA with  $q$  states and  $g$  symbols in the tape alphabet. There are exactly  $qng^n$  distinct configurations of  $M$  for a tape of length  $n$ .*

### Decidable Problems about LBAs

- $A_{\text{LBA}} = \{\langle M, w \rangle \mid M \text{ is an LBA that accepts } w\}$ .

**Theorem 9** (5.9).  $A_{\text{LBA}}$  is decidable.

- $L =$  “On input  $\langle M, w \rangle$ , an encoding of an LBA  $M$  and a string  $w$ :
  1. Simulate  $M$  on input  $w$  for  $qng^n$  steps or until it halts.
  2. If  $M$  has halted, *accept* if it has accepted and reject if it has rejected. If  $M$  has not halted, reject.”

### Undecidable Problems about LBAs

- $E_{\text{LBA}} = \{\langle M \rangle \mid M \text{ is an LBA where } L(M) = \emptyset\}$ .

**Theorem 10** (5.10).  $E_{\text{LBA}}$  is undecidable.

- Assuming that a TM  $R$  decides  $E_{\text{LBA}}$ , we construct a decider  $S$  for  $A_{\text{TM}}$  as follows.
- $S =$  “On input  $\langle M, w \rangle$ , an encoding of a TM  $M$  and a string  $w$ :
  1. Construct an LBA  $B$  from  $\langle M, w \rangle$  that, on input  $x$ , decides whether  $x$  is an accepting computation history for  $M$  on  $w$ .
  2. Run  $R$  on input  $\langle B \rangle$ .
  3. If  $R$  rejects, *accept*; if  $R$  accepts, reject.”

### Undecidable Problems about LBAs (cont.)



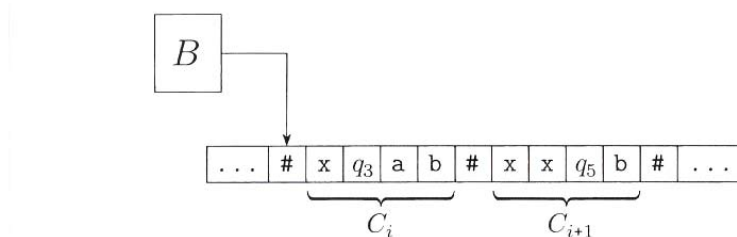
**FIGURE 5.11**  
A possible input to  $B$

Source: [Sipser 2006]

Three conditions of an accepting computation history:

- $C_1$  is the start configuration.
- $C_l$  is an accepting configuration.
- $C_i$  yields  $C_{i+1}$ , for every  $i, 1 \leq i < l$ .

## Undecidable Problems about LBAs (cont.)



**FIGURE 5.12**  
LBA  $B$  checking a TM computation history

Source: [Sipser 2006]

## Undecidable Problems about CFGs

- $ALL_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^* \}$ .

**Theorem 11** (5.13).  $ALL_{CFG}$  is undecidable.

- For a TM  $M$  and an input  $w$ , we construct a CFG  $G$  (by first constructing a PDA) to generate all strings that are *not* accepting computation histories for  $M$  on  $w$ .
- That is,  $G$  generates all strings if and only if  $M$  does not accept  $w$ .
- If  $ALL_{CFG}$  were decidable, then  $A_{TM}$  would be decidable.

## Undecidable Problems about CFGs (cont.)

The PDA for recognizing computation histories that are not accepting works as follows.

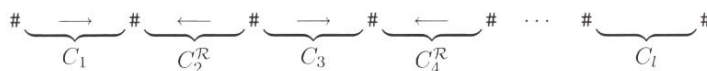
- The input is regarded as a computation history of the form:

$$\#C_1\#C_2^R\#C_3\#C_4^R\#\cdots\#C_l\#$$

where  $C_i^R$  denotes the reverse of  $C_i$ .

- The PDA nondeterministically chooses to check if one of the following conditions holds for the input:
  - $C_1$  is not the start configuration.
  - $C_l$  is not an accepting configuration.
  - $C_i$  does not yield  $C_{i+1}$ , for some  $i$ ,  $1 \leq i < l$ .
- It also accepts an input that is not in the proper form of a computation history.

## Undecidable Problems about CFGs (cont.)



**FIGURE 5.14**  
Every other configuration written in reverse order

Source: [Sipser 2006]

## 4 The Post Correspondence Problem

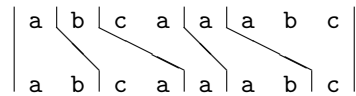
### The Post Correspondence Problem

- Consider a collection of dominos such as follows:

$$\left\{ \left[ \frac{b}{ca} \right], \left[ \frac{a}{ab} \right], \left[ \frac{ca}{a} \right], \left[ \frac{abc}{c} \right] \right\}$$

- A *match* is a list of these dominos (repetitions permitted) where the string of symbols on the top is the same as that on the bottom. Below is a match:

$$\left[ \frac{a}{ab} \right] \left[ \frac{b}{ca} \right] \left[ \frac{ca}{a} \right] \left[ \frac{a}{ab} \right] \left[ \frac{abc}{c} \right]$$



### The Post Correspondence Problem (cont.)

- The Post correspondence problem (PCP) is to determine whether a collection of dominos has a match.
- More formally, an instance of the PCP is a collection of dominos:

$$P = \left\{ \left[ \frac{t_1}{b_1} \right], \left[ \frac{t_2}{b_2} \right], \dots, \left[ \frac{t_k}{b_k} \right] \right\}$$

- A *match* is a sequence  $i_1, i_2, \dots, i_l$  such that  $t_{i_1}t_{i_2}\dots t_{i_l} = b_{i_1}b_{i_2}\dots b_{i_l}$ .
- $PCP = \{ \langle P \rangle \mid P \text{ is an instance of the Post correspondence problem with a match} \}$ .

### Undecidability of the PCP

**Theorem 12** (5.15). *PCP is undecidable*

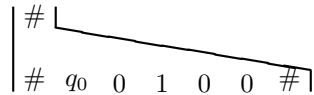
- The proof is by reduction from  $A_{TM}$  via accepting computation histories.
- From any TM  $M$  and input  $w$  we can construct an instance  $P$  where a match is an accepting computation history for  $M$  on  $w$ .
- Assume that a TM  $R$  decides  $PCP$ .
- A decider  $S$  for  $A_{TM}$  constructs an instance of the PCP that has a match if and only if  $M$  accepts  $w$ , as follows.

**Undecidability of the PCP (cont.)**

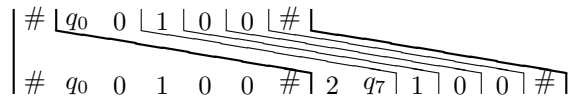
1. Add  $\left[ \frac{\#}{\#q_0w_1w_2 \cdots w_n\#} \right]$  as  $\left[ \frac{t_1}{b_1} \right]$ .
2. For every  $a, b \in \Gamma$  and every  $q, r \in Q$  where  $q \neq q_{\text{reject}}$ ,  
 if  $\delta(q, a) = (r, b, R)$ , add  $\left[ \frac{qa}{br} \right]$ .
3. For every  $a, b, c \in \Gamma$  and every  $q, r \in Q$  where  $q \neq q_{\text{reject}}$ ,  
 if  $\delta(q, a) = (r, b, L)$ , add  $\left[ \frac{cqa}{rcb} \right]$ .
4. For every  $a \in \Gamma$ , add  $\left[ \frac{a}{a} \right]$ .
5. Add  $\left[ \frac{\#}{\#} \right]$  and  $\left[ \frac{\#}{\sqcup\#} \right]$ .

**Undecidability of the PCP (cont.)**

A start configuration (by Part 1):



Suppose  $\delta(q_0, 0) = (q_7, 2, R)$ . With Parts 2-5, the match may be extended to:

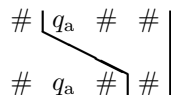


**Undecidability of the PCP (cont.)**

6. For every  $a \in \Gamma$ , add  $\left[ \frac{aq_{\text{accept}}}{q_{\text{accept}}} \right]$  and  $\left[ \frac{q_{\text{accept}}a}{q_{\text{accept}}} \right]$ .



7. Add  $\left[ \frac{q_{\text{accept}}\#\#}{\#} \right]$ .



### Undecidability of the PCP (cont.)

To ensure that a match starts with  $\left[ \frac{t_1}{b_1} \right]$ ,

$S$  converts the collection  $\left\{ \left[ \frac{t_1}{b_1} \right], \left[ \frac{t_2}{b_2} \right], \dots, \left[ \frac{t_k}{b_k} \right] \right\}$  to

$$\left\{ \left[ \frac{\star t_1}{\star b_1 \star} \right], \left[ \frac{\star t_2}{\star b_2 \star} \right], \dots, \left[ \frac{\star t_k}{\star b_k \star} \right], \left[ \frac{\star \diamond}{\star \diamond} \right] \right\}$$

where

$$\begin{aligned} \star u &= \star u_1 \star u_2 \star u_3 \star \dots \star u_n \\ u \star &= u_1 \star u_2 \star u_3 \star \dots \star u_n \star \\ \star u \star &= \star u_1 \star u_2 \star u_3 \star \dots \star u_n \star \end{aligned}$$

## 5 Mapping Reducibility

### Computable Functions

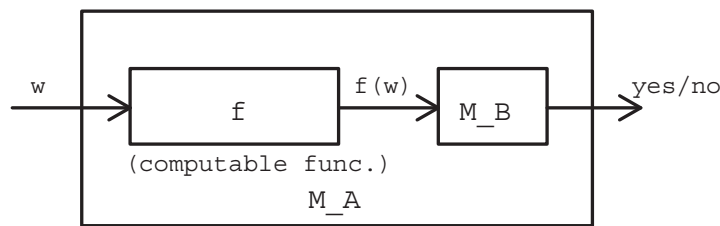
- A Turing machine computes a function by starting with the input to the function on the tape and halting with the output of the function on the tape.

**Definition 13** (5.17). A function  $f : \Sigma^* \rightarrow \Sigma^*$  is a **computable function** if some Turing machine  $M$ , on every input  $w$ , halts with just  $f(w)$  on its tape.

- For example, all usual arithmetic operations on integers are computable functions.
- Computable functions may be transformations of machine descriptions.

### Mapping (Many-One) Reducibility

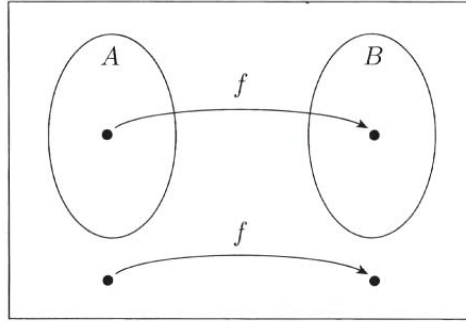
**Definition 14** (5.20). Language  $A$  is **mapping reducible** (many-one reducible) to language  $B$ , written  $A \leq_m B$ , if there is a computable function  $f : \Sigma^* \rightarrow \Sigma^*$ , where for every  $w$ ,  $w \in A \iff f(w) \in B$ .



- This provides a way to convert questions about membership testing in  $A$  to membership testing in  $B$ .

### Mapping (Many-One) Reducibility (cont.)





**FIGURE 5.21**  
Function  $f$  reducing  $A$  to  $B$

Source: [Sipser 2006]

- The function  $f$  is called the *reduction* of  $A$  to  $B$ .

### Reducibility and Decidability

**Theorem 15** (5.22). *If  $A \leq_m B$  and  $B$  is decidable, then  $A$  is decidable.*

- Let  $M$  be a decider for  $B$  and  $f$  a reduction from  $A$  to  $B$ . A decider  $N$  for  $A$  works as follows.
- $N =$  “On input  $w$ :
  1. Compute  $f(w)$ .
  2. Run  $M$  on input  $f(w)$  and output whatever  $M$  outputs.”

**Corollary 16** (5.23). *If  $A \leq_m B$  and  $A$  is undecidable, then  $B$  is undecidable.*

Note:  $(P \wedge Q) \rightarrow R \equiv P \rightarrow (Q \rightarrow R) \equiv P \rightarrow (\neg R \rightarrow \neg Q) \equiv (P \wedge \neg R) \rightarrow \neg Q$

### Reducibility and Decidability (cont.)

**Theorem 17.**  $HALT_{TM}$  is undecidable.

- We show that  $A_{TM} \leq_m HALT_{TM}$ , i.e., a computable function  $f$  exists (as defined by  $F$  below) such that

$$\langle M, w \rangle \in A_{TM} \iff f(\langle M, w \rangle) \in HALT_{TM}.$$

- $F =$  “On input  $\langle M, w \rangle$ :
  1. Construct the following machine  $M'$ .  
 $M' =$  “On input  $x$ :
    - (a) Run  $M$  on  $x$ .
    - (b) If  $M$  accepts, *accept*.
    - (c) If  $M$  rejects, enter a loop.
  2. Output  $\langle M', w \rangle$ .”

## Reducibility and Recognizability

**Theorem 18** (5.28). *If  $A \leq_m B$  and  $B$  is Turing-recognizable, then  $A$  is Turing-recognizable.*

**Corollary 19** (5.29). *If  $A \leq_m B$  and  $A$  is not Turing-recognizable, then  $B$  is not Turing-recognizable.*

**Corollary 20.** *If  $A \leq_m B$  (i.e.,  $\overline{A} \leq_m \overline{B}$ ) and  $A$  is not co-Turing-recognizable, then  $B$  is not co-Turing-recognizable.*

Note: “ $A$  is not co-Turing-recognizable” is the same as “ $\overline{A}$  is not Turing-recognizable”.

## Reducibility and Recognizability (cont.)

**Theorem 21** (5.30 Part One).  *$EQ_{TM}$  is not Turing-recognizable.*

- We show that  $A_{TM}$  reduces to  $\overline{EQ_{TM}}$ , i.e.,  $\overline{A_{TM}}$  reduces to  $EQ_{TM}$ .
- Since  $\overline{A_{TM}}$  is not Turing-recognizable,  $EQ_{TM}$  is not Turing-recognizable.
- $F =$  “On input  $\langle M, w \rangle$ :
  1. Construct the following two machines  $M_1$  and  $M_2$ .  
 $M_1 =$  “On any input: reject.”  
 $M_2 =$  “On any input: Run  $M$  on  $w$ . If it accepts, *accept*.”
  2. Output  $\langle M_1, M_2 \rangle$ .”

## Reducibility and Recognizability (cont.)

**Theorem 22** (5.30 Part Two).  *$EQ_{TM}$  is not co-Turing-recognizable.*

- We show that  $A_{TM}$  reduces to  $EQ_{TM}$ .
- Since  $A_{TM}$  is not co-Turing-recognizable,  $EQ_{TM}$  is not co-Turing-recognizable.
- $G =$  “On input  $\langle M, w \rangle$ :
  1. Construct the following two machines  $M_1$  and  $M_2$ .  
 $M_1 =$  “On any input: *accept*.”  
 $M_2 =$  “On any input: Run  $M$  on  $w$ . If it accepts, *accept*.”
  2. Output  $\langle M_1, M_2 \rangle$ .”