Theory of Computing 2024: Introduction and Preliminaries

(Based on [Sipser 2006, 2013])

Yih-Kuen Tsay

February 19, 2024

1 Overview

What It Is

- The central question:

  What are the fundamental capabilities and limitations of computers?

  /* Similar questions had been asked by theoreticians/mathematicians, even before the first (general-purpose) computer was built. */

- Three main areas:
  - *Automata Theory*
  - *Computability Theory*
  - *Complexity Theory*

Complexity Theory

- Some problems are easy and some hard.

  For example, sorting is easy and scheduling is hard.

  /* A problem is considered (computationally) hard when an efficient (polynomial-time) solution/algorithm to the problem either does not, or is not known to, exist. */

- The central question of complexity theory:

  What makes some problems computationally hard and others easy?

- We don't have the answer to it.

- However, researchers have found a scheme for classifying problems according to their computational difficulty.

- One practical application: cryptography/security.

  /* When breaking an encryption/encipherment involves solving computationally harder problems, one has more confidence in the security of the encryption. */

Dealing with Computationally Hard Problems

Options for dealing with a computationally hard problem:

- Try to simplify it (the hard part of the problem might be unnecessary).
- Settle for an approximate solution.
- Find a solution that usually runs fast.
- Consider alternative types of computation (such as randomized computation).

Computability Theory

- Alan Turing, among other mathematicians, discovered in the 1930s that certain basic problems cannot be solved by computers.
- One example is the problem of determining whether a mathematical statement is true or false.
- Theoretical models of computers developed at that time eventually lead to the construction of actual computers.
- The theories of computability and complexity are closely related.
- *Complexity theory* seeks to classify problems as easy ones and hard ones, while in *computability theory* the classification is by whether the problem is solvable or not.

Automata Theory

- The theories of computability and complexity require a precise, formal definition of a *computer*.
- *Automata theory* deals with the definitions and properties of mathematical models of computation.
- Two basic and practically useful models:
  - *Finite-state*, or simply *finite*, *automaton*
  - *Context-free grammar* (pushdown automaton)

Why You Should Learn the Subject

- It will certainly broaden your knowledge of what computing is fundamentally.
- Below are a few things you may find particularly useful or interesting:
  - Regular expressions, in their original simplest form, for describing patterns of strings/words.
  - Context-free grammars for describing the syntax of a (programming) language.
  - The so-called Turing machines, as the most commonly used model for a computer.
  - Exemplar undecidable problems, which cannot be (perfectly) solved by computers.
  - A proof of SAT being NP-hard, where every NP problem is shown to be polynomially reducible to SAT.

2 Mathematical Notions and Terminology

Sets

- Set, element (member), subset, proper subset
- Multiset

  /* A multiset (or bag) allows for multiple instances of a same element. The sets $\{1,2\}$ and $\{1,1,2\}$, when seen as multisets, are different. */

- Description of a set

  /* There are three common methods for describing a set:

  - Verbal description: the set of all integers from 0 to 99.
  - Enumeration/roster: $\{0, 1, 2, 3, \ldots, 98, 99\}$.
  - Set comprehension: $\{x \in \mathbb{Z} \mid 0 \leq x \leq 99\}$.

  */

- The empty set ($\emptyset$)
- Finite set, infinite set
- Union, intersection, complement
- Power set
- Venn diagram

Sets (cont.)

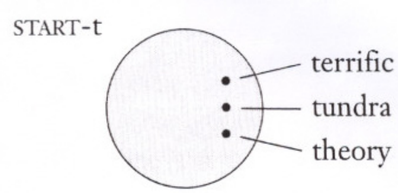


FIGURE 0.1
Venn diagram for the set of English words starting with "t"

Source: [Sipser 2006]

Sets (cont.)

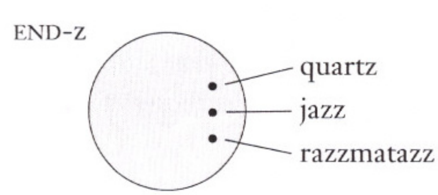


FIGURE 0.2
Venn diagram for the set of English words ending with "z"

Source: [Sipser 2006]

Sets (cont.)

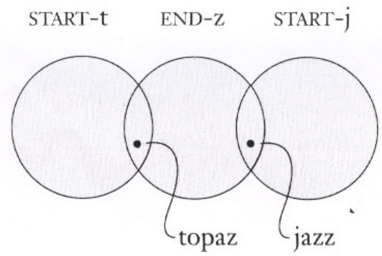


FIGURE **0.3**
Overlapping circles indicate common elements

Source: [Sipser 2006]

Sets (cont.)

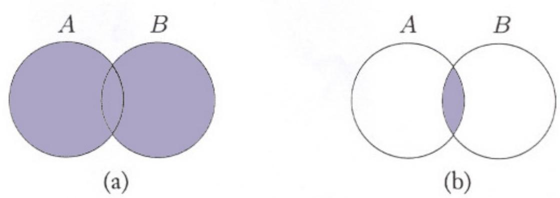


FIGURE **0.4**
Diagrams for (a) $A \cup B$ and (b) $A \cap B$

Source: [Sipser 2006]

Sequences and Tuples

- A *sequence* of objects is a list of these objects in some order. Order is essential and repetition is also allowed.

- Finite sequences are often called *tuples*. A sequence with k elements is a k*-tuple*; a 2-tuple is also called a *pair*.

- The *Cartesian product*, or cross product, of A and B, written as $A \times B$, is the set of all pairs (x, y) such that $x \in A$ and $y \in B$.

- Cartesian products generalize to k sets, A_1, A_2, ..., A_k, written as $A_1 \times A_2 \times \ldots \times A_k$. Every element in the product is a k-tuple.

- A^k is a shorthand for $A \times A \times \ldots \times A$ (k times).

Functions

- A *function* sets up an *input-output* relationship, where the same input always produces the same output.
- If f is a function whose output is b when the input is a, we write $f(a) = b$.
- A function is also called a *mapping*; if $f(a) = b$, we say that f maps a to b.

Functions (cont.)

- The set of possible inputs to a function is called its *domain*; the outputs come from a set called its *range*.
- A function is *onto* if it uses all the elements of the range (it is *one-to-one* if ...).
- The notation $f : D \longrightarrow R$ says that f is a function with domain D and range R.
- More notions and terms: *k-ary function*, *unary function*, *binary function*, *infix notation*, *prefix notation*

Relations

- A *predicate*, or property, is a function whose range is {TRUE,FALSE}.
- A predicate whose domain is $A_1 \times A_2 \times \ldots \times A_k$ is called a k-ary *relation* on $A_1, A_2, \ldots, A_k$. When the A_i's are the same set A, it is simply called a k-ary relation on A.
- A 1-ary relation is usually called a *unary relation* and a 2-ary relation is called a *binary relation*.

Equivalence Relations

- An *equivalence relation* is a special type of binary relation that captures the notion of two objects being *equal* in some sense.
- A binary relation R on A is an equivalence relation if
  1. R is *reflexive* (for every x in A, xRx),
  2. R is *symmetric* (for every x and y in A, xRy if and only if yRx), and
  3. R is *transitive* (for every x, y, and z in A, xRy and yRz implies xRz).

Graphs

- Undirected graph, node (vertex), edge (link), degree
- Description of a graph: $G = (V, E)$
- Labeled graph
- Subgraph, induced subgraph
- Path, simple path, cycle, simple cycle
- Connected graph
- Tree, root, leaf
- Directed graph, outdegree, indegree
- Strongly connected graph

Graphs (cont.)

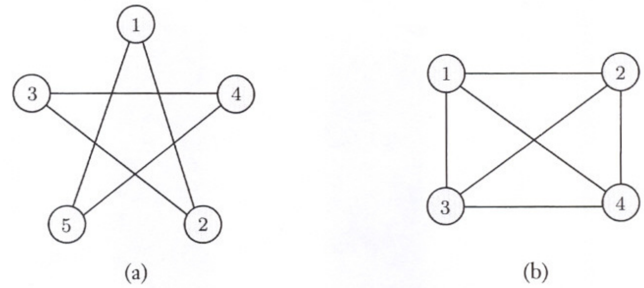


FIGURE **0.12**
Examples of graphs

Source: [Sipser 2006]

Graphs (cont.)

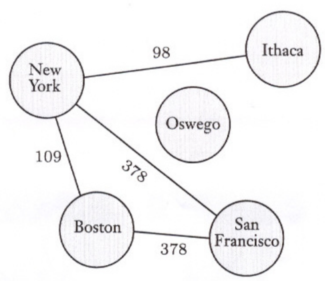


FIGURE **0.13**
Cheapest nonstop air fares between various cities

Source: [Sipser 2006]

Graphs (cont.)

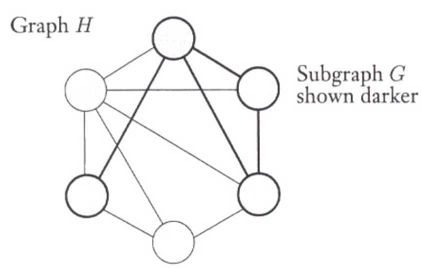


FIGURE **0.14**
Graph *G* (shown darker) is a subgraph of *H*

Source: [Sipser 2006]

Graphs (cont.)

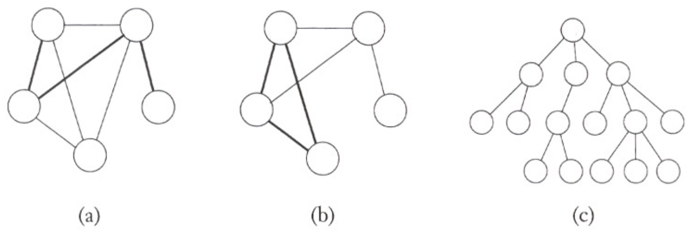


FIGURE 0.15
(a) A path in a graph, (b) a cycle in a graph, and (c) a tree

Source: [Sipser 2006]

Graphs (cont.)

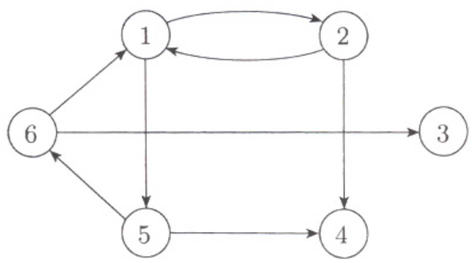


FIGURE 0.16
A directed graph

Source: [Sipser 2006]

Graphs (cont.)

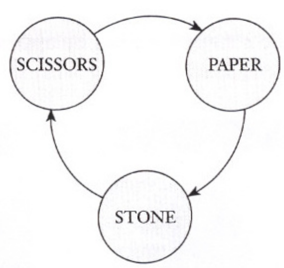


FIGURE 0.18
The graph of the relation *beats*

Source: [Sipser 2006]

Strings and Languages

- An *alphabet* is any finite set of *symbols*.
- A *string* over an alphabet is a finite sequence of symbols from that alphabet.
- The *length* of a string w, written as $|w|$, is the number of symbols that w contains.
- The string of length 0 is called the *empty string*, written as ε.
- The *concatenation* of x and y, written as xy, is the string obtained from appending y to the end of x.
- A *language* is a set of strings.
- More notions and terms: *reverse*, *substring*, *lexicographic ordering*.

Boolean Logic

- *Boolean logic* is a mathematical system built around the two *Boolean values* TRUE (1) and FALSE (0).
- Boolean values can be manipulated with *Boolean operations*: *negation* or NOT ($\neg$), *conjunction* or AND ($\wedge$), *disjunction* or OR ($\vee$).

$$\begin{array}{lll} 0 \wedge 0 \triangleq 0 & 0 \vee 0 \triangleq 0 & \neg 0 \triangleq 1 \\ 0 \wedge 1 \triangleq 0 & 0 \vee 1 \triangleq 1 & \neg 1 \triangleq 0 \\ 1 \wedge 0 \triangleq 0 & 1 \vee 0 \triangleq 1 & \\ 1 \wedge 1 \triangleq 1 & 1 \vee 1 \triangleq 1 & \end{array}$$

- Unknown Boolean values are represented symbolically by *Boolean variables* or *propositions*, e.g., P, Q, etc.

Boolean Logic (cont.)

- Additional Boolean operations: *exclusive or* or XOR ($\oplus$), *equality/equivalence* ($\leftrightarrow$ or $\equiv$), *implication* ($\rightarrow$).

$$\begin{array}{lll} 0 \oplus 0 \triangleq 0 & 0 \leftrightarrow 0 \triangleq 1 & 0 \rightarrow 0 \triangleq 1 \\ 0 \oplus 1 \triangleq 1 & 0 \leftrightarrow 1 \triangleq 0 & 0 \rightarrow 1 \triangleq 1 \\ 1 \oplus 0 \triangleq 1 & 1 \leftrightarrow 0 \triangleq 0 & 1 \rightarrow 0 \triangleq 0 \\ 1 \oplus 1 \triangleq 0 & 1 \leftrightarrow 1 \triangleq 1 & 1 \rightarrow 1 \triangleq 1 \end{array}$$

- All in terms of conjunction and negation:

$$\begin{array}{lcl} P \vee Q & \equiv & \neg(\neg P \wedge \neg Q) \\ P \rightarrow Q & \equiv & \neg P \vee Q \\ P \leftrightarrow Q & \equiv & (P \rightarrow Q) \wedge (Q \rightarrow P) \\ P \oplus Q & \equiv & \neg(P \leftrightarrow Q) \end{array}$$

Logical Equivalences and Laws

- Two logical expressions/formulae are *equivalent* if each of them implies the other, i.e., they have the same truth value.
- Equivalence plays a role analogous to equality in algebra.
- Some laws of Boolean logic:
  - (Distributive) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
  - (Distributive) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  - (De Morgan's) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
  - (De Morgan's) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

3 Definitions, Theorems, and Proofs

Definitions, Theorems, and Proofs

- *Definitions* describe the objects and notions that we use. Precision is essential to any definition.
- After we have defined various objects and notions, we usually make *mathematical statements* about them. Again, the statements must be precise.
- A *proof* is a convincing logical argument that a statement is true. The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof.
- A *theorem* is a mathematical statement proven true. *Lemmas* are proven statements for assisting the proof of another more significant statement.
- *Corollaries* are statements seen to follow easily from other proven ones.

Finding Proofs

- Find proofs isn't always easy; no one has a recipe for it.
- Below are some helpful general strategies:
  1. Carefully read the statement you want to prove.
  2. Rewrite the statement in your own words.
  3. Break it down and consider each part separately. For example, $P \iff Q$ consists of two parts: $P \to Q$ (the forward direction) and $Q \to P$ (the reverse direction).
  4. Try to get an intuitive feeling of why it should be true.

Tips for Producing a Proof

- A well-written proof is a sequence of statements, wherein each one follows by simple reasoning from previous statements in the sequence.
- Tips for producing a proof:
  - *Be patient.* Finding proofs takes time.
  - *Come back to it.* Look over the statement, think about it, leave it, and then return some time later.
  - *Be neat.* Use simple, clear text and/or pictures; make it easy for others to understand.
  - *Be concise.* Emphasize high-level ideas, but be sure to include enough details of reasoning.

An Example Proof

Theorem 1. *For any two sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.*

Proof. We show that every element of $\overline{A \cup B}$ is also an element of $\overline{A} \cap \overline{B}$ and vice versa.

Forward ($x \in \overline{A \cup B} \rightarrow x \in \overline{A} \cap \overline{B}$):

	$x \in \overline{A \cup B}$	
$\rightarrow$	$x \notin A \cup B$	, def. of complement
$\rightarrow$	$x \notin A$ and $x \notin B$	, def. of union
$\rightarrow$	$x \in \overline{A}$ and $x \in \overline{B}$	, def. of complement
$\rightarrow$	$x \in \overline{A} \cap \overline{B}$	, def. of intersection

Reverse ($x \in \overline{A} \cap \overline{B} \rightarrow x \in \overline{A \cup B}$): ...

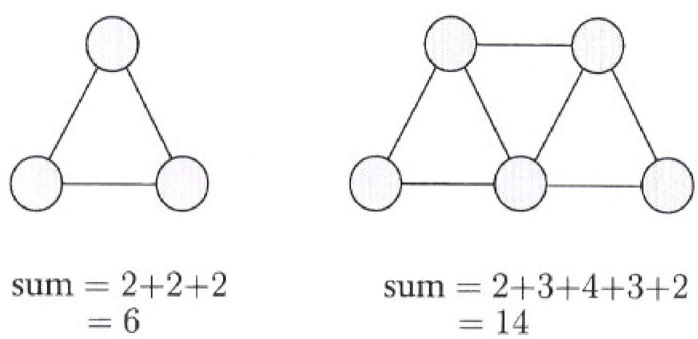
Another Example Proof

Theorem 2. *In any graph G, the sum of the degrees of the nodes of G is an even number.*

Proof.

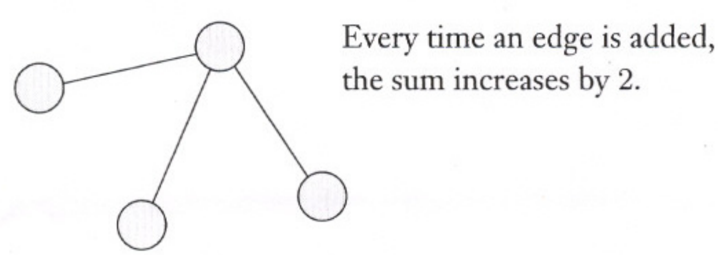
- Every edge in G connects two nodes, contributing 1 to the degree of each.
- Therefore, each edge contributes 2 to the sum of the degrees of all the nodes.
- If G has e edges, then the sum of the degrees of the nodes of G is $2e$, which is even.

Another Example Proof (cont.)



Source: [Sipser 2006]

Another Example Proof (cont.)



Source: [Sipser 2006]

4 Types of Proof

Types of Proof

- *Proof by construction*: prove that a particular type of object exists, by showing how to construct the object.
- *Proof by contradiction*: prove a statement by first assuming that the statement is false and then showing that the assumption leads to an obviously false consequence, called a contradiction.
- *Proof by induction*: prove that all elements of an infinite set have a specified property, by exploiting the inductive structure of the set.

Proof by Construction

Theorem 3. *For each even number n greater than 2, there exists a 3-regular graph with n nodes.*

Proof. Construct a graph $G = (V, E)$ with n $(= 2k > 2)$ nodes as follows.

Let V be $\{0, 1, \ldots, n-1\}$ and E be defined as

$$\begin{aligned} E \;=\; & \{\{i, i+1\} \mid \text{for } 0 \le i \le n-2\} \cup \\ & \{\{n-1, 0\}\} \cup \\ & \{\{i, i+n/2\} \mid \text{for } 0 \le i \le n/2-1\}. \end{aligned}$$

Proof by Contradiction

Theorem 4. $\sqrt{2}$ *is irrational.*

Proof. Assume toward a contradiction that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = \frac{m}{n}$ for some integers m and n, which *cannot both be even.*

$\sqrt{2} = \frac{m}{n}$	, from the assumption
$n\sqrt{2} = m$	, multipl. both sides by n
$2n^2 = m^2$	, square both sides
m is even	, m^2 is even
$2n^2 = (2k)^2 = 4k^2$	, from the above two
$n^2 = 2k^2$	, divide both sides by 2
n is even	, n^2 is even

Now both m and n are even, a contradiction.

Example: Home Mortgages

P: the *principle* (amount of the original loan).
I: the yearly *interest rate*.
Y: the monthly payment.
M: the *monthly* multiplier $= 1 + I/12$.
P_t: the amount of loan outstanding after the t-th month; $P_0 = P$ and $P_{k+1} = P_k M - Y$.

Theorem 5. *For each $t \ge 0$,*

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right).$$

Proof by Induction

Theorem 6. *For each $t \geq 0$,*

$$P_t = PM^t - Y(\frac{M^t - 1}{M - 1}).$$

Proof. The proof is by induction on t.

- *Basis*: When $t = 0$, $PM^0 - Y(\frac{M^0-1}{M-1}) = P = P_0$.

Proof by Induction (cont.)

- *Induction step*: When $t = k + 1$ ($k \geq 0$),

$$\begin{array}{cl} & P_{k+1} \\ = & \quad \{\text{definition of } P_t\} \\ & P_k M - Y \\ = & \quad \{\text{the induction hypothesis}\} \\ & (PM^k - Y(\frac{M^k-1}{M-1}))M - Y \\ = & \quad \{\text{distribute } M \text{ and rewrite } Y\} \\ & PM^{k+1} - Y(\frac{M^{k+1}-M}{M-1}) - Y(\frac{M-1}{M-1}) \\ = & \quad \{\text{combine the last two terms}\} \\ & PM^{k+1} - Y(\frac{M^{k+1}-1}{M-1}) \end{array}$$

Structural Induction

- Structural induction is a generalization of mathematical induction on the natural numbers.
- It is used to prove that some proposition $P(x)$ holds for all x of some sort of recursively/inductively defined structure such as binary trees.
- Proof by structural induction:
  1. Base case: the proposition holds for all the minimal structures.
  2. Inductive step: if the proposition holds for the immediate substructures of a certain structure S, then it also holds for S.